# Algebras and modules II

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The following notes are written for the seminar *representation theory of algebras* at university of Zurich in the spring semester of 2020, supervised by Dr. D. Tubbenhauer at the mathematical institute. I will follow closely the book *Quiver representations* by Ralf Schiffler (Springer, 2014).

## introduction

**an aside** Although the seminar is about quiver representations, most of the definitions and results in my part can be introduced in the more general setting of abstract algebra. I will therefore follow this approach in the first part of my notes. Naturally, at the end of the notes I will present - in the context of quivers - examples of the results previously given.

**preliminaries** I will assume knowledge of the following algebraic structures: groups, rings and their ideals, fields, the notion of an algebraically closed field as well as the concept of morphisms between such structures. Anything else that is needed and not mentioned here will be defined properly later on.

**main goal** The corollary that I will be working towards makes a statement about the relation between finite-dimensional modules over $k$-algebras (where $k$ is an algebraically closed field) and the algebra that arises from the endomorphisms on those modules. More precisely, the relation is concerned with the question of decomposition of modules into direct sums of submodules on the one hand, and the maximal ideals of the endomorphism-algebra on the other hand. Without further ado, I will now present the necessary tools to arrive at this result.

## algebras and modules

Throughout, let $k$ be an algebraically closed field.

**definition 1.** *A $k$-algebra $A$ is a ring with $1$, equipped with a $k$-vector space structure, such that*

1. *the vector space inherits addition from the ring (i.e. addition in the ring and in the vector space coincide)*
2. *scalar multiplication in the vector space is compatible with the multiplication of the ring, i.e. for $c \in k$ and $r, s \in A$ one has*

$$c(rs) = (cr)s = r(cs) = (rs)c \tag{1}$$

Alternatively, one could regard a $k$-algebra as a ring $A$ together with a ring homomorphism $\nu : k \to Z(A)$, where $Z(A)$ denotes the center of $A$ (i.e. the subring consisting of those elements commuting with every element in $A$). Scalar multiplication $\mu$ of the vector space can then be described in terms of $\nu$ by

$$\begin{aligned} \mu : k \times A &\to A \\ (c, r) &\mapsto \nu(c)r \end{aligned} \tag{2}$$

From another perspective, one could also define a $k$-algebra to be a vector space equipped with a bilinear map. These definitions are equivalent, as one can intuitively see by realizing that the bilinear map specifies the multiplication in the ring, and the bilinearity guarantees point 2 in the above definition 1.

**definition 2.** *Let $R$ be a ring with $1$. A right $R$-module $M$ is an abelian group $(M, +)$ together with a binary operation $M \times R \to M$ such that, for $m_1, m_2 \in M$ and $r_1, r_2 \in R$, one has*

1. $(m_1 + m_2)r_1 = m_1 r_1 + m_2 r_1$
2. $m_1(r_1 + r_2) = m_1 r_1 + m_2 r_2$
3. $m_1(r_1 r_2) = (m_1 r_1) r_2$
4. $m_1 1 = m_1$

Let $\mathcal{G} = \{m_1, \dots m_k\} \subseteq M$ for some (right $R$-)module $M$. Then $M$ is said to be generated by $\mathcal{G}$, if, for every $m \in M$, there exist elements $r_1, \dots, r_k \in R$ such that $m = m_1 r_1 + \dots + m_k r_k$. Consequently, $M$ is called finitely generated, if $|\mathcal{G}| < \infty$.

**definition 3.** *Let $M$ and $N$ be $R$-modules. A morphism from $M$ to $N$ is a map preserving the structures of addition and multiplication. More precise, a morphism $h : M \to N$ satisfies*

1. $h(m + m') = h(m) + h(m')$
2. $h(mr) = h(m)r$

*for all $m, m' \in M$ and $r \in R$.*

An endomorphism of modules is a morphism from $M$ to $M$. The set of all endomorphisms for a given $A$-module $M$ over a $k$-algebra $A$, denoted by $\mathrm{End}(M)$, has the structure of a $k$-algebra: the structure of the underlying vector space is given by addition and scalar multiplication of endomorphisms. That is, for $g, h \in \mathrm{End}(M), c \in k$ and $m \in M$

$$\begin{aligned} (g + h)(m) &= g(m) + h(m) \\ (g \cdot c)(m) &= g(mc) \end{aligned} \tag{3}$$

Multiplication in $\mathrm{End}(M)$ can be defined as composition of endomorphisms, i.e.

$$(g \cdot h)(m) = g \circ h(m) \tag{4}$$

**definition 4.** *Let $R$ be a ring. A left respectively right ideal $I$ in $R$ is called maximal, if, whenever there exists an ideal $J$ such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$. The intersection of all maximal right (resp. left) ideals is called the right (resp. left) Jacobson radical of $R$, denoted by* $\mathrm{rad}\, R$.

It is a fact (that I will state without proof) that the *right radical* coincides with the *left radical*, thus I will simply write *radical* in what follows.

**lemma 5.** *Let $R$ be a ring with unity and $a \in R$. Then $a \in \mathrm{rad}\, R \implies 1 - ba$ has a two-sided inverse for every $b \in R$.*

*Proof.* Let $a \in \operatorname{rad} R$ and assume $1 - ba$ has no left inverse for some $b \in R$. It must then be contained in some maximal left ideal $J \subset R$ (it is clearly contained in some left ideal, namely in the ideal $(1 - ba)$ generated by itself, and since it has no left-inverse by assumption, $(1 - ba) \neq R$). Since $J$ is maximal and the radical of $R$ is the intersection of all maximal left ideals, we further have that $\operatorname{rad} R \subset J$. Since by assumption $a \in \operatorname{rad} R$, also $a \in J$ and so $ba \in J$ for every $b \in R$. But then $1 - ba + ba = 1 \in J$ contradicting $J \neq R$. Therefore $1 - ba$ has a left-inverse in $R$. I will denote this left-inverse by $\kappa$. Then

$$\kappa(1 - ba) = 1 \implies \kappa = 1 + \kappa ba = 1 + (\kappa b)a \tag{5}$$

But we have just shown that the element $1 + (\kappa b)a = 1 - (-\kappa b)a$ has a left-inverse, say $l$. Therefore

$$\begin{aligned} 1 = l\kappa &= l(1 + (\kappa b)a) \\ &= l + l\kappa ba \\ &= l + ba \end{aligned} \tag{6}$$

Hence

$$l = 1 - ba \tag{7}$$

and so

$$1 = l\kappa = (1 - ba)\kappa \tag{8}$$

which shows that $\kappa$ is also a right-inverse for $1 - ba$. ■

The statement is in fact stronger: it is an if and only if statement that also holds true for the element $1 - ab$. Although I will need the other direction of the implication later on, I decided to only give the proof for this direction, as it would go beyond the scope of this work to present every proof. Nevertheless, I would like to point out that a proof of the fact that the right Jacobson radical equals the left Jacobson radical can be done by showing that $1 - ab$ has a two-sided inverse if and only if $1 - ba$ has a two-sided inverse for every $b$, together with the reverse direction of lemma 5.

If $I$ is a right ideal in the ring $R$ and $M$ is an $R$-module, then the set $MI = \{m_1 i_1 + \dots m_t i_t : m_j \in M, i_j \in R\}$ is a submodule of M. This notation is needed for the following

**lemma 6.** *Let $I$ be a two-sided ideal in a ring $R$ (that is, a left- and right-ideal) such that $I$ is contained in the radical of $R$, and let $M$ be a finitely generated $R$-module. Then $MI = M$ implies $M = 0$.*

*Proof.* The proof will be done by induction on the cardinality of the generating set of $M$. Suppose $M$ is generated by the set $\{m_1, \dots, m_s\}$ and let $MI = M$.

If $s = 1$, then $M = m_1 R$. Since by assumption $M = MI$, this implies that $\forall m \in M$ there exist elements $\mu_1, \dots \mu_t \in M$ and $i_1, \dots i_t \in I$ such that

$$m = \mu_1 i_1 + \cdots + \mu_t i_t \tag{9}$$

Thus in particular

$$m_1 = \mu_1 i_1 + \cdots + \mu_t i_t \tag{10}$$

Also, since $M = m_1 R$, one has that for every $j \in \{1, ..., t\}$ there are elements $r_j \in R$ such that

$$\mu_j = m_1 r_j \tag{11}$$

Therefore,

$$\begin{aligned} m_1 &= \mu_1 i_1 + \cdots + \mu_t i_t \\ &= m_1 r_1 i_1 + \cdots + m_1 r_t i_t \\ &= m_1 (r_1 i_1 + \cdots + r_t i_t) \\ &= m_1 \zeta \end{aligned} \tag{12}$$

In other words

$$m_1(1 - \zeta) = 0 \tag{13}$$

Since $r_j \in R$ and $i_j \in I$ for $1 \leq j \leq t$, we find $\zeta \in I$. By the assumption of the lemma, the ideal $I$ is contained in the radical of $R$. Therefore, by lemma 5, we know that for every $s \in R$ the element $1 - s\zeta$ has a two-sided inverse in $R$. In particular (by setting $s = 1$), $1 - \zeta$ has a two sided inverse in $R$. I will call this inverse $\kappa$. Then

$$m_1 = m_1 1 = m_1[(1 - \zeta)\kappa] = [m_1(1 - \zeta)]\kappa = 0\kappa = 0 \tag{14}$$

and since $m_1$ generates $M$, the conclusion $M = 0$ follows.

Now assume the statement holds true for some $s$. Let $M$ be generated by $\mathcal{G} = \{m_1, \dots, m_{s+1}\}$. As before, $M = MI$ implies

$$m_1 = \mu_1 i_1 + \cdots + \mu_t i_t \tag{15}$$

for some $\mu_j \in M$ and $i_j \in I$. For every $k \in \{1, \dots, t\}$, we can write $\mu_k$ in terms of elements of $\mathcal{G}$ as

$$\mu_k = m_1 r_{1,k} + \cdots + m_{s+1} r_{s+1,k} \tag{16}$$

with $r_{j,k} \in R$ for $1 \leq j \leq s+1$. Putting the last two equations together, we find

$$\begin{aligned} m_1 &= (m_1 r_{1,1} + \cdots + m_{s+1} r_{s+1,1}) i_1 + \cdots + (m_1 r_{1,t} + \cdots + m_{s+1} r_{s+1,t}) i_t \\ &= m_1 r_{1,1} i_1 + \cdots + m_{s+1} r_{s+1,1} i_1 + \cdots + m_1 r_{1,t} i_t + \cdots + m_{s+1} r_{s+1,t} i_t \\ &= m_1 (r_{1,1} i_1 + \cdots + r_{1,t} i_t) + E = m_1 \Theta + E \end{aligned} \tag{17}$$

where

$$E = m_2 r_{2,1} i_1 + \cdots + m_{s+1} r_{s+1,1} i_1 + \cdots + m_2 r_{2,t} i_t + \cdots + m_{s+1} r_{s+1,t} i_t \tag{18}$$

By equation 17 we find

$$m_1 (1 - \Theta) = E \tag{19}$$

and, since $\Theta = (r_{1,1} i_1 + \cdots + r_{1,t} i_t) \in I \subset \operatorname{rad} R$, by lemma 5 it follows that $1 - \Theta$ has a two-sided inverse. Again, I will call this inverse $\kappa$. Then

$$m_1 = m_1 (1 - \Theta) \kappa = E \kappa \tag{20}$$

Looking at the form of $E$ in equation 18, we see that, since $\kappa \in I \subset R$, $E\kappa$ can be written as a $R$-linear combination of the elements in the set $\{m_2, \dots, m_{s+1}\}$. That is, $m_1$ is redundant in the generating set $\mathcal{G}$, and so $M$ is generated by a set of cardinality $s$. By induction hypothesis it thus follows that $M = 0$ as desired. ■

As a consequence, the radical of a finite dimensional algebra is nilpotent, as the following corollary shows. Recall that an element $r$ of a ring is called nilpotent, if there exists $n \in \mathbb{N}$ such that $r^n = 0$. Analogously, an ideal $I$ of a ring is called nilpotent, if there exists $n \in \mathbb{N}$ such that $I^n = 0$, where $I^n = \{\sum_{j=1}^{s} i_{j,1} \dots i_{j,n} : i_{j,k} \in I\}$. If $I$ is an ideal, so is $I^n$ for all $n$.

**corollary 7.** *Let $A$ be a finite dimensional algebra. Then* $\operatorname{rad} A$ *is nilpotent.*

*Proof.* Since ideals of an algebra are subgroups of the additive group and are closed under multiplication with arbitrary elements of the ring, as well as closed under scalar multiplication, every ideal of an algebra is a subalgebra. It therefore follows that if $A$ is finite dimensional, so is every ideal in $A$. This implies that for a decreasing chain of ideals $A \supseteq I_0 \supseteq I_1 \supseteq \dots$ in $A$, at some point the dimension cannot be reduced anymore and thus the chain becomes stationary, i.e. there exists $n \in \mathbb{N}$ such that for every $m > n$, $I_m = I_n$. Now for a two-sided ideal $I$ in $A$, $I^n \supseteq I^{n+1}$ holds true for any $n$. These considerations constitute the following chain

$$A \supseteq \operatorname{rad} A \supseteq (\operatorname{rad} A)^2 \supseteq \dots \tag{21}$$

that leads to the identity

$$(\operatorname{rad} A)^m = (\operatorname{rad} A)^n \tag{22}$$

for some $n \in \mathbb{N}$ and every $m > n$. In particular for $m = n + 1$

$$(\operatorname{rad} A)^n \operatorname{rad} A = (\operatorname{rad} A)^n \tag{23}$$

Recall that every ideal in $A$ is finite-dimensional, meaning it has a finite basis as a $k$-vector space. One can thus view every ideal in $A$ as a finitely generated $A$-module. By identifying $(\operatorname{rad} A)^n = M$ and $\operatorname{rad} A = I$, all conditions for lemma 6 are met, hence

$$(\operatorname{rad} A)^n = 0 \tag{24}$$

This completes the proof that the radical of $A$ is nilpotent. ■

**definition 8.** *An algebra $A$ is said to be local, if it has a unique maximal right ideal.*

Recall that the left and right radical coincide, and the right radical is the intersection of all maximal right ideals. So for a local algebra, the unique maximal right ideal is the same as the radical (and therefore also the same as the unique maximal left ideal).

**lemma 9.** *Let $A$ be a $k$-algebra. If, for every $a \in A$, either $a$ or $1 - a$ is invertible, then $A$ is local.*

*Proof.* Assume that for every $a \in A$ either $a$ or $1 - a$ has an inverse in $A$. Consider an element $a$ that is not in the radical of $A$. The remark after the proof of lemma 5 implies that there exists $b \in A$ such that $1-ab$ has no inverse in $A$. By assumption, $ab$ thus has a two-sided inverse in $A$. Say $ab\tau = 1$. But then $b\tau$ is an inverse of $a$. Since $a \in A - \operatorname{rad} A$, this shows that $A / \operatorname{rad} A$ is a field. This however implies that $\operatorname{rad} A$ is maximal. Indeed, assume there exists an ideal $J$ containing $\operatorname{rad} A$. Consider an element $r \in J - \operatorname{rad} A$. Since $r$ is not in the radical, it has an inverse in $A$, due to the considerations above. Thus also $1 \in J$ which implies $J = A$. Finally, because the radical is maximal, it is the unique maximal ideal, so $A$ is local as desired. ∎

The converse ($A$ local implies either $a$ or $1-a$ invertible for every $a \in A$) also holds true. As was done for lemma 5, I restrict this proof to only one direction. However, I will use the other direction for corollary 11 as well as for examples at the very end of this work.

**definition 10.** *Let $A$ be a $k$-algebra. $a \in A$ is called idempotent, if $a^2 = a$.*

**corollary 11.** *If $A$ is local, the only idempotents are $0$ and $1$.*

*Proof.* Let $A$ be local and $e \in A$ be idempotent. Then $e^2 = e \implies e(1-e) = 0$. By the remark above, either $e$ or $1-e$ has an inverse. Assume $\tau e = 1$. Then $0 = \tau 0 = \tau e(1-e) = 1-e$, hence $e = 1$. On the other hand, if $(1-e)\tau = 1$, then $0 = 0\tau = e(1-e)\tau = e$ and so $e = 0$. ∎

**definition 12.** *Let $A$ be a $k$-algebra and $M_1, \dots M_s$ be $A$-modules. The direct sum $M_1 \oplus \cdots \oplus M_s$ is the $A$-module whose vector space is the direct sum of the vector spaces of the $M_i$ and whose module structure is given by*

$$(m_1, \dots, m_s)a = (m_1 a, \dots, m_s a) \tag{25}$$

*A module is called indecomposable, if it can not be written as a direct sum of two proper submodules.*

I am now prepared to state the main result as mentioned in the introduction.

**corollary 13.** *Let $A$ be a $k$-algebra. Let $M$ be a finite dimensional $A$-module and $\operatorname{End} M$ its endomorphism algebra. The following are equivalent.*

1. *$M$ is indecomposable*
2. *Every $\phi \in \operatorname{End} M$ is of the form $\phi = \lambda \operatorname{id} + \theta$, where $\theta \in \operatorname{End} M$ is nilpotent and $\lambda \in k$*

*3.* End $M$ *is local*

*Proof.* "1 ⇒ 2": Let $M$ be indecomposable, and consider $\phi \in \operatorname{End} M$. Since $M$ is a finite dimensional $A$-module, $\phi$ is a $k$-linear map between finite dimensional $k$-vector spaces. In particular, we can use the notion of the characteristic polynomial $\chi_\phi$ of $\phi$. Furthermore, since $k$ is algebraically closed, $\chi_\phi$ splits into linear factors over $k$, hence we can write

$$\chi_\phi(x) = \prod_{i=1}^{t} (x - \lambda_i)^{\nu_i} \tag{26}$$

where $\lambda_i$ are the eigenvalues of $\phi$ with corresponding multiplicity $\nu_i$, hence $\dim M = \sum_{i=1}^{t} \nu_i$. Let

$$\begin{aligned} h_i &= (\phi - \lambda_i \operatorname{id})^{\nu_i} = \phi^{\nu_i} + a_{\nu_i - 1}\phi^{\nu-1} + \cdots + a_1\phi + a_0 \operatorname{id} \\ M_i &= \ker h_i \end{aligned} \tag{27}$$

$h_i$, being a linear combination of powers of $\phi$, is itself an element of the endomorphism algebra $\operatorname{End} M$. Hence, $M_i$ being the kernel of an endomorphism of $A$-modules is itself an $A$-module. Observe that $M_i \cap M_j = \emptyset \ \forall i \neq j$ and $\dim M_i = \nu_i$. Therefore $M$ can be decomposed into a direct sum

$$M = M_1 \oplus \cdots \oplus M_t \tag{28}$$

But by assumption $M$ is indecomposable, forcing $t = 1$. In other words, $\phi$ has only one eigenvalue. From linear algebra we know that we can find a basis such that $\phi$ is in Jordan normal form (this is possible because the characteristic polynomial splits into linear factors over $k$). Thus we can write

$$\phi = \lambda \operatorname{id} + \theta \tag{29}$$

where

$$\theta_{ij} = \begin{cases} 1, \text{ if } j = i+1 \\ 0 \text{ else} \end{cases} \tag{30}$$

and so $\theta^2 = 0$, i.e. in particular nilpotent.

"2 ⇒ 3": Using lemma 9, if $\phi$ is invertible, then $\operatorname{End} M$ is local. So assume $\phi$ is not invertible. This means that $\lambda = 0$, and so $\phi = \theta$. Our goal is to show that $\operatorname{id} - \phi$ is invertible, because then lemma 9 implies again that

End $M$ is local. There are two ways to do so:
1. Since $\phi = \theta$, $\phi^2 = 0$. Therefore $\mathrm{id} = \mathrm{id} - \phi^2 = (\mathrm{id} + \phi)(\mathrm{id} - \phi)$.
2. Consider the matrix representation of $\mathrm{id} - \theta$. Denote by $\psi_n$ the $n \times n$-matrix that has the form of $\mathrm{id} - \theta$. Using Laplace's formula for the determinant, one has

$$\det \psi_n = \det \psi_{n-1} - \det Q + \sum_{i=1}^{n-2} (-1)^{i+1} \cdot 0 \cdot \det \mathcal{M}_i \tag{31}$$

where the $\mathcal{M}_i$ are some $(n-1) \times (n-1)$ matrices and $Q$ is a matrix whose first column is zero. Thus $\det Q = 0$ and so by iteration $\det \psi_n = 1 \forall n$, since $\psi_1 = 1$. Hence $\mathrm{id} - \theta$ is invertible.

"3 $\Rightarrow$ 1": Let End $M$ be local and assume that $M = M_1 \oplus M_2$. Let $\pi_i : M \to M_i$ and $\iota_i : M_i \to M$ be the natural projections and inclusions respectively. Then $\iota_i \circ \pi_i \in \mathrm{End}\, M$ is idempotent, and so by corollary 11 it is either zero or the identity on $M$, since End $M$ is local. If $\iota_i \circ \pi_i$ is zero, then $M_i = 0$, because $\iota_i \circ \pi_i(m) = (0, 0)$ for all $m = (m_1, m_2) \in M$ implies $\iota_i(m_i) = 0$ for all $m_i \in M_i$ and so $M_i = 0$. On the other hand, if $\iota_i \circ \pi_i$ is the identity on $M$, then $\iota_i \circ \pi_i(m) = m$ implies $\iota_i(m_i) = (m_1, m_2)$. This however shows that $m_j = 0$ for $j \neq i$, since $\iota_i$ maps $m_i$ into either $(m_1, 0)$ or $(0, m_2)$. Therefore $M_j = 0$ and so in both cases $M$ is indecomposable. This completes the proof. ■

## quiver representations

I will assume basic knowledge about quivers, in particular the definition of a quiver, a representation and a path. For notational purposes, recall that a quiver $Q = (Q_0, Q_1, s, t)$ consists of a set of vertices $Q_0$, a set of arrows $Q_1$ and maps $s, t : Q_1 \to Q_0$, mapping arrows to their starting and ending points respectively.

For $i \in Q_0$ and $\alpha \in Q_1$, I will denote the collection of $k$-vector spaces of a representation by $M_i$ and the collection of $k$-linear maps by $\phi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$.

For $i, j \in Q_0$, $\alpha_k \in Q_1$ and $l \in \mathbb{N}$, a path of length $l$ from $i$ to $j$ will be denoted by $c = (i|\alpha_1, \ldots, \alpha_l|j)$. Of course, this definition only makes sense if $t(\alpha_k) = s(\alpha_{k+1})$ for every $0 < k < l$.

Let $c = (i|\alpha_1, \dots, \alpha_l|j)$ and $c' = (j|\alpha'_1, \dots, \alpha'_{l'}|k)$ be paths. The concatenation $c \cdot c'$ of those paths is defined to be $c \cdot c' = (i|\alpha_1, \dots, \alpha_l, \alpha'_1, \dots, \alpha'_{l'}|k)$.

**definition 14.** *Let $Q$ be a quiver and $c, c'$ as before. Let $V(Q)$ be the $k$-vector space having as basis all paths in $Q$. If one enriches the structure of $V(Q)$ with a multiplication defined on the basis elements by $cc' = c \cdot c'$ if $s(\alpha'_1) = t(\alpha_l)$ and zero otherwise, the resulting object is called the path algebra and denoted by $kQ$.*

Recall the notation of the constant path $e_i = (i||i)$. Furthermore, for simplicity of notation, assume that $s$ and $t$ act on paths as if they were single arrows, i.e. for $c = (i|\alpha_1, \dots, \alpha_l|j)$ define $s(c) = i$ and $t(c) = j$.

**lemma 15.** *$1 \in kQ$ is given by $1 = \sum_{i \in Q_0} e_i$.*

*Proof.* Let $p = \sum_c \lambda_c c$ be any element in $kQ$, where $\lambda_c \in k$. Then

$$\begin{aligned} 1p &= \sum_{i \in Q_0} e_i \sum_c \lambda_c c = \sum_{i \in Q_0} \sum_c \lambda_c e_i c \\ &= \sum_{i \in Q_0} \sum_c \lambda_c c \delta_{is(c)} = \sum_{i \in Q_0} \sum_{c : s(c) = i} \lambda_c c \\ &= \sum_c \lambda_c c = p \end{aligned} \tag{32}$$

where $\delta_{ij}$ denotes the kronecker delta. $p1 = p$ can be proven in a similar vein, by swapping sums and replacing $s(c)$ with $t(c)$. ∎

**definition 16.** *A path of the form $c = (i|\alpha|i)$ is called a loop.*

As an example, consider the quiver that consists of a single loop. That is, $Q = (\{1\}, \{\alpha\}, s, t)$ with $s(\alpha) = t(\alpha) = 1$. Its path algebra $kQ$ is isomorphic to $k[x]$, the algebra of polynomials in one variable over $k$. To see this, look at the basis $\mathcal{B}$ of $kQ$. It consists by definition of all paths in $Q$, so $\mathcal{B} = \{e_1, \alpha, \alpha^2, \dots\}$. Multiplication is given by summing the turns around the loop, in other words summation of the exponents $\alpha^s \alpha^t = \alpha^{s+t}$, with the convention $\alpha^0 = e_1$. An explicit isomorphism $\varphi : kQ \xrightarrow{\sim} k[x]$ defined on the basis $\mathcal{B}$ is therefore given by $\varphi(\alpha^t) = x^t$. Note that (with the exception of $e_1$) there are no multiplicative inverses. This reflects the property that the loop has an orientation given by the arrow ("going the other way round is not possible"). Motivated by these considerations, the more general definition follows immediately.

**definition 17.** *A path of the form $c = (i|\alpha_1, \dots \alpha_l|i)$ is called an oriented cycle.*

A loop is thus an oriented cycle of length 1.

**corollary 18.** *Let $Q$ be a quiver without oriented cycles. Then the ideal generated by all arrows is equal to the radical of $kQ$.*

Remark: For the proof of this corollary I will need finiteness of the set $Q_0$ of vertices (finiteness of $Q_0$ also forces finiteness of $Q_1$). I am not sure if the corollary would hold true for e.g. the quiver $Q = (\mathbb{N}, \{\alpha_i\}_{i\in\mathbb{N}}, [\alpha_i \mapsto i], [\alpha_i \mapsto i+1])$. I will only give a sketch of the proof, as it would require some more work to be able to give every detail. The reader is referred to *Quiver representations* by Ralf Schiffler for an exact proof.

*Proof.* (sketch) Let $I_Q$ be the two-sided ideal of $kQ$ generated by all arrows in the quiver $Q$. Since there are no oriented cycles by assumption, and $Q_0$ is finite, there is a longest path (not necessarily unique). Call it's length $l^{max}$. Because there is no longer path than this, there is no possibility to concatenate $l^{max}+1$ consecutive arrows. Thus the product of $l^{max}+1$ elements (i.e. arrows) in $kQ$ must be zero. Hence $I_Q^{l^{max}+1} = 0$, or in other words $I_Q$ is nilpotent. Further, if $I$ is a two-sided nilpotent ideal in a ring $R$, then $I \subset \operatorname{rad} R$. Indeed, let $n$ be such that $I^n = 0$. Let $i \in I$. Then, for every $r \in R$, $ri \in I$ and so $(ri)^n = 0$. Therefore

$$1 = 1 - (ri)^n = (1 + ri + (ri)^2 + \dots + (ri)^{n-1})(1 - ri) \tag{33}$$

This shows that $1 - ri$ has an inverse for every $r \in R$, and so by the remark after the proof of lemma 5, $i \in \operatorname{rad} R$, hence $I \subset \operatorname{rad} R$, as $i$ was arbitrary. As a first conclusion, this shows that $I_Q \subset \operatorname{rad} kQ$.
Furthermore, for any ideal $I$ in an algebra $A$, if $I$ is two-sided and nilpotent and the algebra $A/I$ is isomorphic to the direct product of some copies of the underlying field $k$, then $I \supset \operatorname{rad} A$. One can show that $kQ/I_Q \cong \prod_{i\in Q_0} k$, and so $I_Q \supset \operatorname{rad} kQ$. The conclusion $I_Q = \operatorname{rad} kQ$ thus follows. ∎

I will end these lines by presenting an example of an endomorphism algebra consisting of endomorphisms of a representation of a quiver. The aim is to make use of corollary 13 in a concrete situation.

Let $Q = (\{1,2\}, \{\alpha, \beta\}, s(\alpha) = s(\beta) = 1, t(\alpha) = t(\beta) = 2)$. Let $M_1 = M_2 = k^2$, $\phi_\alpha = \mathbb{1}$ and $\phi_\beta = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$. An endomorphism $f = (f_i)_{i \in \{1,2\}} : M \to M$ must by definition satisfy $f_2 \circ \phi_i = \phi_i \circ f_1$ for $i \in \{\alpha, \beta\}$. Since $\phi_\alpha$ is the identity on $k^2$, this implies that $f_1 = f_2$. For simplicity, in what follows I will denote $f_1 = f_2$ simply by $F$ (so that with the above notation $f = (f_i)_{i \in \{1,2\}} = (F, F)$). Using $F \in M_2(k)$, write $F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some $a, b, c, d \in k$. Looking at the second commuting square, i.e. the case $i = \beta$, it follows

$$\begin{aligned} F \circ \phi_\beta &= \phi_\beta \circ F \\ \implies \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \implies \begin{bmatrix} a & \lambda a + b \\ c & \lambda c + d \end{bmatrix} &= \begin{bmatrix} a + \lambda c & b + \lambda d \\ c & d \end{bmatrix} \end{aligned} \tag{34}$$

Therefore $a = a + \lambda c$, or $\lambda c = 0$. Also $\lambda a + b = \lambda d + b$, or $\lambda a = \lambda d$. I will proceed by distinguishing cases.

<u>case 1:</u> $\lambda \neq 0$

Then $\lambda c = 0$ implies $c = 0$ and $\lambda a = \lambda d$ implies $a = d$. Hence $F = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ and the endomorphism algebra of the representation $M$ consists of

$$\operatorname{End} M = \{m \in M_2(k) | m_{ij} = a\delta_{ij} + b\delta_{i(i+1)}\} \tag{35}$$

Now if $a \neq 0$, then $\det F = a^2$ and so $F$ is invertible. If $a = 0$, then $\mathbb{1} - F = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}$ is invertible, since $\det F = 1$. Thus for every $F \in \operatorname{End} M$, either $F$ or $\mathbb{1} - F$ is invertible, and so by lemma 9, $\operatorname{End} M$ is local. Using corollary 13, this shows that $M$ is indecomposable.

<u>case 2:</u> $\lambda = 0$

In this case $\phi_\beta = \phi_\alpha = \mathbb{1}$, and so there is no constraint on $f_1 = f_2 = F$.

Hence $\operatorname{End} M = M_2(k)$. The elements $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbb{1} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are both not invertible. Therefore, by the remark after the proof of lemma 9, $M$ can not be local.