Morphisms of representations

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October 9, 2019

These notes are written for the seminar *representation theory of finite groups* at University of Zurich, Fall Semester 2019, supervised by Dr. D. Tubbenhauer. All definitions, proofs and theorems are taken from the book *Representation Theory of Finite Groups* by B. Steinberg, Springer New York, 2012.

Throughout the notes, G is a finite group. U, V and W are complex vector spaces and ϕ, ψ and ρ are representations of G. The notation $\phi_g := \phi(g)$ is used for the image of a group element under the representation.

definition *Let $\phi : G \to GL(V)$, $\rho : G \to GL(W)$ be representations of a group G. A morphism from ϕ to ρ is a linear map $T : V \to W$ such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "\phi_g"] \arrow[d, "T"] & V \arrow[d, "T"] \\ W \arrow[r, "\rho_g"] & W \end{tikzcd}$$

i.e. $T \circ \phi_g = \rho_g \circ T$ for all $g \in G$.

The set of all morphisms from ϕ to ρ is denoted by $Hom_G(\phi, \rho)$. Sometimes such a morphism is called *intertwiner*, or *intertwining operator*. We will stick to the name *morphism*.

remark *If T is an isomorphism (of vector spaces), then $\phi \sim \rho$ by definition. Furthermore $Hom_G(\phi, \rho) \subseteq Hom(V, W)$.*

One can easily see that composing two morphisms of representations results again in a morphism of representations. To be more precise, we will prove the following

theorem *Let $\phi : G \to GL(U)$, $\psi : G \to GL(V)$ and $\rho : G \to GL(W)$ be representations of a group G. Suppose $T \in Hom_G(\phi, \psi)$ and $S \in Hom_G(\psi, \rho)$. Then $ST \in Hom_G(\phi, \rho)$.*

Informally, the idea is to extend the above diagram and "to flip the path twice along the diagonal of the small squares", by using the fact that for each "sub-square" we have a free choice of path.

$$\begin{tikzcd}
U \arrow[r, "T"] \arrow[d, "\phi_g"] & V \arrow[r, "S"] \arrow[d, "\psi_g"] & W \arrow[d, "\rho_g"] \\
U \arrow[r, "T"] & V \arrow[r, "S"] & W
\end{tikzcd}$$

Proof. $T \in Hom_G(\phi,\psi) \implies T\phi_g = \psi_g T$ and $S \in Hom_G(\psi,\rho) \implies S\psi_g = \rho_g S$, hence $ST\phi_g = S\psi_g T = \rho_g ST$ as desired. □

remark *One can furthermore show that $Hom_G(\phi,\psi)$ admits the structure of a vector space, thus being a subspace of $Hom(V,W)$. Indeed, for T_1,$T_2 \in Hom_G(\phi,\psi)$ and c_2,$c_2 \in \mathbb{C}$ (this fact even holds for general vector spaces over some field. Actually, everything said so far need not be over complex vector spaces, but holds in more generality), we find $(c_1T_1 + c_2T_2)\phi_g = c_1T_1\phi_g + c_2T_2\phi_g = c_1\psi_g T_1 + c_2\psi_g T_2 = \psi_g(c_1T_1 + c_2T_2)$.*

Recall the definition of an irreducible representation

definition *A representation $\phi : G \to GL(V)$ is said to be irreducible, if there is no non-trivial proper G-invariant subspace (i.e. the only G-invariant subspaces of V are $\{0\}$ and V).*

Morphisms of irreducible representations are of a special kind. *Schur's Lemma* states that there are only two possibilities for such morphisms. In order to prove Schur's lemma, we need the following

proposition *Let $T \in Hom_G(\phi,\rho)$ for representations $\phi : G \to GL(V)$ and $\psi : G \to GL(W)$. Then $\ker T$ and $\operatorname{Im} T$ are G-invariant subspaces of V and W respectively.*

Proof. Let $v \in \ker T$, $w \in \operatorname{Im} T$ and $g \in G$. Then $T\phi_g v = \psi_g T v = 0$, since $v \in \ker T$. Thus $\phi_g v \in \ker T$. Now set $w = Tu$ for some $u \in V$. Then $\psi_g w = \psi_g T u = T\phi_g u \in \operatorname{Im} T$. □

Schur's lemma *Let ϕ, ρ be irreducible representations of a group G, and $T \in Hom_G(\phi,\rho)$. If $T \neq 0$, then T is invertible.*

Proof. Let $0 \neq T \in Hom_G(\phi,\rho)$. By the above proposition, $\ker T$ is a G-invariant subspace of V. But ϕ being irreducible implies that the only G-invariant subspaces of V are V and $\{0\}$. Hence $\ker T = \{0\}$, since otherwise $T = 0$. We therefore have that T is injective.
On the other hand we know that $\operatorname{Im} T$ is a G-invariant subspace of W. Since ψ is irreducible, it follows that $\operatorname{Im} T = W$, since otherwise $\operatorname{Im} T = \{0\}$, i.e. $T = 0$. This shows surjectivity and hence completes the proof. □

There are two consequences that follow from Schur's lemma. One quite easily, namely if $\phi \nsim \psi$, then $Hom_G(\phi,\psi) = 0$. For if $Hom_G(\phi,\psi) \neq 0$, there would exist an invertible $0 \neq T \in Hom_G(\phi,\psi)$, and so $\phi \sim \psi$. The other one will be stated as a "sub-lemma", since its proof requires some more work.

sub-lemma *If $\phi = \psi$, then $T = \lambda I$, where $\lambda \in \mathbb{C}$*

Proof. Let λ be an eigenvalue of T. This is there first time that we actually need the fact that the vector spaces are complex, in order to guarantee the existence of at least one eigenvalue (this need not be true for some generic field). By definition of eigenvalues, we know that $T-\lambda I$ is not invertible (since $T-\lambda I$ maps every v to zero). Using $I \in Hom_G(\phi,\phi)$ and the fact that $Hom_G(\cdot,\cdot)$ has the structure of a vector space, we know that $T-\lambda I \in Hom_G(\phi,\phi)$. But by Schur's lemma, $T-\lambda I = 0$, since it is not invertible. Therefore $T = \lambda I$ as desired. $\square$

With this preliminary work in mind, we can now proof a corollary about the classification of irreducible representations of abelian groups.

corollary *Let G be an abelian group, and $\phi : G \to GL(V)$ an irreducible representation. Then $deg(\phi) = 1$.*

Proof. Consider some $h \in G$, and denote $T = \phi_h$. Then $T \in Hom_G(\phi,\phi)$, i.e.

$$\begin{array}{ccc} V & \xrightarrow{\phi_g} & V \\ \downarrow T & & \downarrow T \\ V & \xrightarrow{\phi_g} & V \end{array}$$

commutes. This is true since $T\phi_g = \phi_h\phi_g = \phi_{hg} = \phi_{gh} = \phi_g T$, where we used for the second equation the fact that ϕ is a representation, and for the third equation the commutativity in G. By the "sub-lemma" it follows that $T = \lambda I$, or in this case $\phi_h = \lambda_h I$. We will now show that for any non-zero $v \in V$, $span(v)$ is a G-invariant subspace of V. This will then show that $V = span(v)$ by irreducibility of ϕ, and therefore $dim(V) = 1 = deg(\phi)$. To achieve this, consider some $0 \neq v \in V$ and $k \in \mathbb{C}$. We then have that $\phi_h(kv) = \lambda_h I kv = \lambda_h kv \in span(v)$. Since h was arbitrary, $span(v)$ is a G-invariant subspace. $\square$

We will now discuss some applications of these results to linear algebra. Before we proceed, recall *Mascke's theroem* that states that every representation of a finite group is completely reducible (the proof was given in the last presentation).

corollary *Let $\phi : G \to GL_n(\mathbb{C})$ be a representation of a finite abelian group G. Then there exists an invertible matrix T such that $D = T^{-1}\phi_g T$ is diagonal for all $g \in G$.*

Proof. By Maschke's theorem we know that ϕ is completely reducible. Hence we can write $\phi \sim \phi^{(1)} \bigoplus \cdots \bigoplus \phi^{(r)}$ with $\phi^{(i)}$ irreducible for all $i \in \{1,...,r\}$. Furthermore, by the corollary above we have that $deg\left(\phi^{(i)}\right) = 1 \forall i \in \{1,...,r\}$. Since the dimension of the direct sum of vector spaces is the same as the sum of the dimensions of the individual vector spaces, we can conclude that $r = n$. As a consequence (of $deg\phi^{(i)} = 1$), every $\phi_g^{(i)}$ is a non-zero complex number for all $g \in G$. Let T be the isomorphism for $\phi \sim \bigoplus_{1\le i\le r} \phi^{(i)}$. Since direct sums of representations in terms of matrices are the individual matrices as block-matrices along the diagonal, $T^{-1}\phi_g T$ is diagonal:

$$T^{-1}\phi_g T = \begin{bmatrix} \phi_g^{(1)} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \phi_g^{(2)} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & \phi_g^{(2)} \end{bmatrix}$$

$\square$

corollary *Let $A \in GL_m(\mathbb{C})$ be of finite order, i.e. there exist $n \in \mathbb{N}$ such that $A^n = I_n$. Then A is diagonalizable. Moreover, the eigenvalues of A are nth-roots of unity.*

Proof. Define the representation $\phi : \mathbb{Z}/n\mathbb{Z} \to GL_n(\mathbb{C})$ by sending the equivalence class of $z \in \mathbb{Z}$ to A^z. This is indeed a representation, since $\phi([z_1] + [z_2]) = A^{z_1+z_2} = A^{z_1}A^{z_2}$, and it is well-defined as for $z \to z + n$ we find $\phi([z+n]) = A^{z+n} = A^z A^n = A^z = \phi([z+n])$. So by the corollary before, there exists $T \in GL_m(\mathbb{C})$ such that $D = T^{-1}AT$ is diagonal, with the eigenvalues on the diagonal. This proves the first part of the corollary.
For the second statement, consider $D^n = (T^{-1}AT)^n = T^{-1}IT = I$. Now since D^n is still diagonal, with entries on the diagonal being nth-powers of the entries on the diagonal of D, we find the relations $D_{ii}^n = 1$ for all $1 \leq i \leq m$. So the eigenvalues D_{ii} are nth-roots of unity. $\square$