Seminar talk: Maschke's Theorem

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Outline

In this talk, we will proceed along the following lines:

- We will first define what a unitary representation is. In order to speak of unitary representation, the corresponding vector space must be an inner product space. This is not a restriction in our case since we always work with vector spaces over the complex numbers. And any vector space over the field of complex numbers can be equipped with an inner product.
- We will show that any unitary representation of a finite group is either irreducible or decomposable. (This does not hold for infinite groups.)
- We will show that any representation of a finite group is equivalent to a unitary representation. It follows that any representation of a finite group is either irreducible or decomposable.
- Then, we are ready to prove the central theorem of this chapter (Maschke): every representation of a finite group is completely reducible. (In the next chapter, it will turn out that the decomposition into irreducible representations is unique.)

Talk

For this seminar, G is always a finite group (unless otherwise stated).

Definition 1. Let V be an inner product space. A representation $\varphi : G \longrightarrow GL(V)$ is called unitary if φ_g is unitary for every $g \in G$, i.e.

$$\langle \varphi_g v, \varphi_g w \rangle = \langle v, w \rangle \ \forall v, w \in V.$$

Example 2. We ask ourselves which elements in $GL_1(\mathbb{C})$ are unitary. Let $u \in GL_1(\mathbb{C})$, i.e. $u(z) := \alpha z$. For u to be unitary, the following must hold for all $x, y \in \mathbf{C}$:

$$\begin{aligned}\langle u(x), u(y)\rangle = \langle x, y\rangle &\iff \alpha\overline{\alpha}x\overline{y} = x\overline{y}\\ &\iff \alpha\overline{\alpha} = 1\\ &\iff |\alpha| = 1\end{aligned}$$

I.e. The only unitary linear maps in $GL_1(\mathbb{C})$ are the multiplications with a complex number on the unit circle.

Example 3. This example shows a unitary representation of the (infinite) additive group of the real numbers $\mathbb{R}$. Let $\varphi : \mathbb{R} \longrightarrow GL_1(\mathbb{C})$ with $\varphi_t z := e^{2\pi i t} z$. This representation is unitary, since

$$\begin{aligned}\langle \varphi_t z_1, \varphi_t z_2\rangle &= e^{2\pi i t} z_1 \overline{e^{2\pi i t} z_2}\\ &= e^{2\pi i t} e^{-2\pi i t} z_1 \overline{z_2}\\ &= \langle z_1, z_2\rangle\end{aligned}$$

Proposition 4. *Every unitary representation of a group G is either irreducible or decomposable.*

Proof. Let φ be a representation of G in V. If φ is irreducible, the proposition is obviously true. Thus, let φ be non-irreducible (reducible). Then, there is a non-zero proper G-invariant subspace $W \subset V$. Its orthogonal complement (the set of all elements in V which are orthogonal to any element in W) is also non-zero and $V = W \oplus W^\perp$.

It only remains to prove that $W^\perp$ is also G-invariant. Let $w \in W$ and $w^\perp \in W^\perp$. We must show that $\varphi_g w^\perp \in W^\perp$, i.e. $\varphi_g w^\perp$ is orthogonal to any element in W.

$$\begin{aligned}\left\langle \varphi_g w^\perp, w\right\rangle &\overset{\varphi \text{ unitary}}{=} \left\langle \varphi_{g^{-1}}\varphi_g w^\perp, \varphi_{g^{-1}} w\right\rangle\\ &= \left\langle w^\perp, \varphi_{g^{-1}} w\right\rangle\\ &= 0\end{aligned}$$

The last step follows since $w^\perp \in W^\perp$ and $\varphi_{g^{-1}} w \in W$. This completes the proof. ☐

Proposition 5. *Every representation of a finite group is equivalent to a unitary representation.*

The proof of the above proposition is along the following lines:

- For any representation φ, we construct another representation $\rho : G \longrightarrow GL_n(\mathbb{C})$ with a special inner product.
- We show that this representation is equivalent to φ and unitary with respect to the inner product defined above.

Proof. Let $\varphi : G \longrightarrow GL(V)$ be an n-dimensional representation of G. Let B be a basis of V and $T : V \longrightarrow \mathbb{C}^n$ the isomorphism that yields the coordinates of a vector in V with respect to B. Define $\rho : G \longrightarrow GL_n(\mathbb{C})$ such that $\rho_g := T\varphi_g T^{-1}$. I.e. ρ_g takes a corrdinate vector in $\mathbb{C}^n$, transforms it to the corresponding element in V, applies the original representation φ_g, and transforms the result in V back to to its coordinate vector in $\mathbb{C}^n$.

It is easy to show that ρ is indeed a representation since φ itself is a representation. We compute

$$\begin{aligned}\rho_{g_1+g_2} &= T\varphi_{g_1+g_2}T^{-1}\\ &= T\varphi_{g_1}\varphi_{g_2}T^{-1}\\ &= T\varphi_{g_1}T^{-1}T\varphi_{g_2}T^{-1}\\ &= \rho_{g_1}\rho_{g_2}\end{aligned}$$

and

$$\begin{aligned}\rho_{g^{-1}} &= T\varphi_{g^{-1}}T^{-1}\\ &= T\varphi_g^{-1}T^{-1}\\ &= ((T\varphi_g^{-1}T^{-1})^{-1})^{-1}\\ &= (T\varphi_g T^{-1})^{-1}\\ &= \rho_g^{-1}\end{aligned}$$

Furthermore, the representation is equivalent to φ by construction.

Based on the inner product on V we define an inner product on $\mathbb{C}^n$ as follows:

$$(v, w) := \sum_{g\in G} \langle \rho_g v, \rho_g w\rangle\,.$$

We show that $(\cdot,\cdot)$ is indeed an inner product. We compute

$$\begin{aligned}(c_1v_1 + c_2v_2, w) &= \sum_{g\in G} \langle \rho_g(c_1v_1 + c_2v_2), \rho_g w\rangle\\ &= \sum_{g\in G} (c_1 \langle \rho_g v_1, \rho_g w\rangle + c_2 \langle \rho_g v_2, \rho_g w\rangle)\\ &= c_1 \sum_{g\in G} \langle \rho_g v_1, \rho_g w\rangle + c_2 \sum_{g\in G} \langle \rho_g v_2, \rho_g w\rangle\\ &= c_1(v_1, w) + c_2(v_2, w).\end{aligned}$$

Next, we verify

$$\begin{aligned}(v, w) &= \sum_{g\in G} \langle \rho_g v, \rho_g w\rangle\\ &= \sum_{g\in G} \overline{\langle \rho_g w, \rho_g v\rangle}\\ &= \overline{(w, v)}.\end{aligned}$$

Next, we show that $(\cdot,\cdot)$ is positive definite. We must thus show that $(v,v) \geq 0 \,\forall v \in V$ and $(v,v) = 0 \Longleftrightarrow v = 0$. The first relation is obvious since

$$(v,v) = \sum_{g \in G} \langle \rho_g v, \rho_g v \rangle \,.$$

The sum is positive since we are adding only positive numbers. The second relation is proven as follows: $(v,v) = 0 = \sum_{g\in G} \langle \rho_g v, \rho_g v\rangle \Rightarrow \langle \rho_g v, \rho_g w\rangle = 0 \,\forall g \in G$ since we are adding non-negative numbers. Specially, $\langle \rho_1 v, \rho_1 v\rangle = \langle v, v\rangle = 0 \Rightarrow v = 0$. Thus, we have finally established that $(\cdot,\cdot)$ is indeed an inner product. It remains to show that ρ is unitary with respect to this inner product. To show this, we compute

$$\begin{aligned}
(\rho_h v, \rho_h w) &= \sum_{g\in G} \langle \rho_g \rho_h v, \rho_g \rho_h w\rangle \\
&= \sum_{g\in G} \langle \rho_{gh} v, \rho_{gh} w\rangle \\
&= \sum_{x\in G} \langle \rho_x v, \rho_x w\rangle \\
&= (v,w).
\end{aligned}$$

To verify the change of variables above, one must show that the summation over x still runs through all elements of G. I.e. the map $\tau_h : G \longrightarrow G$ with $\tau_h(g) = hg$ must be bijective for any $h \in G$. Injectivity follows like so: $\tau_h(g) = \tau_h(g') \Leftrightarrow hg = hg' \Leftrightarrow g = g'$. To proof surjectivity, it must be shown that $\forall g \in G \;\exists g' \in G : \tau_h(g') = g$. Such a $g' \in G$ indeed exists, namely $g' = h^{-1}g$. Indeed: $\tau_h(g') = hg' = hh^{-1}g = g$. This completes the proof. □

Corollary 6. *Any non-zero representation φ of a finite group is either irreducible or decomposable.*

Proof. Any representation φ of a finite group is equivalent to a unitary representation ρ which is either irreducible or decomposable. The irreducibility and decomposability, however, are "invariant" under equivalence of representations. This means that also φ is either irreducible or decomposable. □

Example 7. In this example we present a representation of the infinite group $\mathbb{Z}$ which is not irreducible but still not decomposable. Define $\varphi : \mathbb{Z} \longrightarrow GL_2(\mathbb{C})$ by

$$\varphi(n) := \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

We show that this is a representation by calculating

$$\begin{aligned} \varphi(m+n) &= \begin{bmatrix} 1 & m+n \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \\ &= \varphi(m)\varphi(n) \end{aligned}$$

and

$$\begin{aligned} \varphi(-n) &= \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \varphi^{-1}(n). \end{aligned}$$

The vector $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\varphi(n)$ for all $n \in \mathbb{Z}$:

$$\varphi(n)e_1 = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, the space spanned by this eigenvector is a $\mathbb{Z}$-invariant subspace. Explicitly, for $e \in \text{span}(e_1)$:

$$\varphi(n)e = \varphi(n)\lambda e_1 = \lambda\varphi(n)e_1 = \lambda e_1 \in \text{span}(e_1).$$

I.e. φ is not irreducible. On the other hand, if φ were decomposable, it would be a direct sum of two one-dimensional representations. Such a representation is diagonal. However, e.g. $\varphi(1)$ is not diagonalizable, as the characteristical polynomial $\det(x1 - \varphi(1))$ has a double root at $x = 1$. Therefore, φ is not decomposable.

Theorem 8. *Every representation of a finite group is completely reducible.*

Proof. The proof works by induction over the dimension of the representation. Let thus φ be a representation of a finite group G on V. For $\dim(V) = 1$, φ is already irreducible since any one dimensional vector space cannot have any one-dimensional proper subspaces. Let then the statement be true for $\dim(V) = n$ and φ be a representation of dimension $n+1$. If φ is already irreducible, then we are done. If φ is not irreducible, then it is decomposable according to Corollary 9. This means that $V = V_1 \oplus V_2$ with V_1 and V_2 being G-invariant subspaces. Since both $\dim(V_1)$ and $\dim(V_2)$ are smaller than $n+1$, the representations $\varphi_{|V_1}$ and $\varphi_{|V_2}$ are completely reducible. This means that $V_1 = U_1 \oplus \cdots \oplus U_s$ and $V_2 = W_1 \oplus \cdots \oplus W_r$ where U_i and W_j are all G-invariant subspaces and the representations $\varphi_{|U_i}$ and $\varphi_{|W_j}$ are all irreducible for $i \in \{1, ..., s\}$ and $j \in \{1, ..., r\}$. Then, $V = U_1 \oplus \cdots \oplus U_s \oplus W_1 \oplus \cdots \oplus W_r$ and thus φ is completely reducible. $\square$

Remark 9. There is an analogy to the spectral theorem in linear algebra. The spectral theorem states that any Hermitian (self adjoint) matrix is diagonalizable. More specifically, let V be a complex inner product vector space of dimension n. A square matrix A is called Hermitian, if $A = A^* \equiv \overline{A^T}$. This is equivalent to saying that $\langle v, Aw \rangle = \langle Av, w \rangle$ for all $v, w \in V$. Diagonalizability means that there exists a unitary matrix U (i.e. a matrix with $U^* = U^{-1}$) and a diagonal matrix Λ such that $A = U\Lambda U^*$. The proof of the spectral theorem also works by induction over the dimension of the matrix.

There is, however, a difference between complete reducibility of a representation and the spectral theorem. In representation theory, complete reducibility does not mean that all sub-representations are one-dimensional. However, the spectral theorem states that all eigenspaces are one-dimensional.