Talk, Kioloa, 18.Feb.2019

Who coloured my Dynkin diagram?

0 ADE

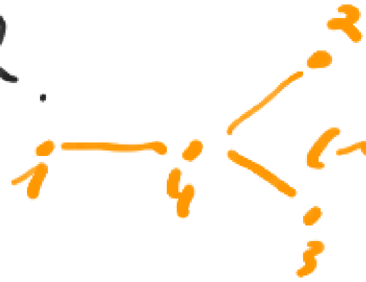
Chebyshev Polynomial

$$XU_{\ell+1}(X) = U_{\ell+2}(X) + U_\ell(X)$$

$+\ U_0 = 1 \qquad U_1 = X$

"Minimal polynomial for alg. integers in $]-2,2[$"

Let Γ be a loopless, connected finite graph.



$$\longmapsto A(\Gamma) = \left(\begin{array}{ccc|c} & 0 & & 1 \\ & & & 1 \\ & & & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}\right)$$

Problem: Classify all Γ such that $U_{\ell+1}(\Gamma) = 0$

Aside: Why is this a reason-able question?

Example: $X^2 = 0 \rightsquigarrow$ Solutions all 2-nilpotent matrices $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\rightsquigarrow$ too many? No bound on: Size of matrices or entries.

Solution (Frobenius)

Restrict to $A(\Gamma)$'s $\rightsquigarrow$ only $\boxed{\text{finitely}}$ many solutions?

Solution (Smith ~ 1969)

ADE graphs for $e+2$ being the Coxeter number (at most)

A_m $\bullet - \bullet \cdots \bullet - \bullet$ ✓ $e = m-1$

D_m ✓ $e = 2m-4$

D_m		✓	$e = 2m-1$
E_6		✓	$e = 12$
E_7		✓	$e = 18$
E_8		✓	$e = 30$

Aside: One extra family of solutions if condition "loopless" is removed. Tadpoles

T_m:  ✓ $e = 2m+1$

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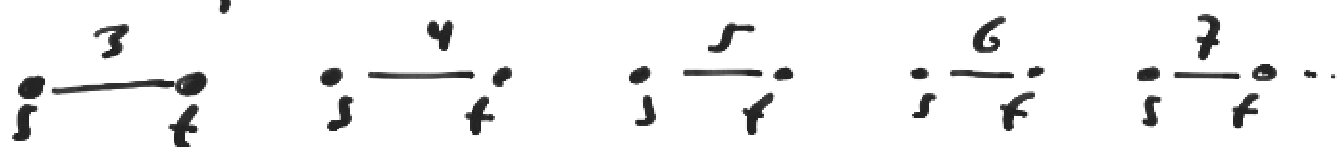
① Classical rep. theory (some...)

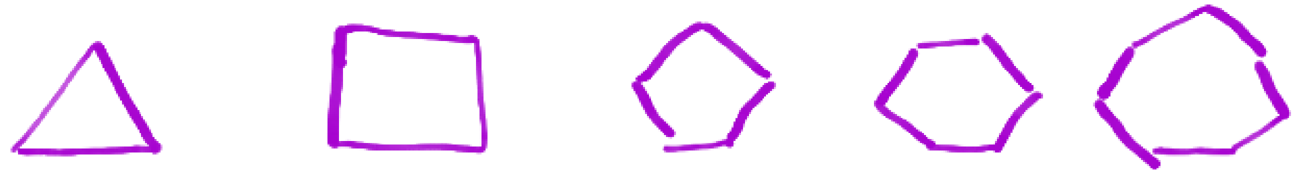
$$W_{e+2} = \langle s,t \mid s^2 = t^2 = 1, \underbrace{sts\ldots}_{e+2} = tst\ldots \rangle$$

↖ dihedral group of order $2e+4$

⇝ Symmetry groups of regular $e+2$-gons

Example:





A_2 | B_2 | H_2 | G_2 | $I_2(7)$

$\rightsquigarrow$ simplices | cubes | "Dodecahedra" | X | X

$\hookrightarrow$ finite Coxeter groups

A_n — simplex

$B_n = C_n$ — cubes

D_n — semi cubes

$I_2(n)$ — regular polygon

$+ F_4$ "24-cell"

$+ G_2$ sixgon

$+ E_6, E_7, E_8$

$+ H_2, H_3, H_4$ "Dodecahedra"

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reps of $\mathbb{C}[W_{e+2}]$ or $H_v(I_2(e+2))$

1-dim $\quad s \mapsto \lambda_s \in \mathbb{C} \quad t \mapsto \lambda_t \in \mathbb{C}$

$s^2 = t^2 \rightsquigarrow \lambda = \pm 1 \qquad M_{\lambda_s, \lambda_t}$

$e \equiv 0 \mod 2 \qquad\qquad e \equiv 1 \mod 2$

$M_{1,1}$, $M_{1,-1}$, $M_{-1,1}$, $M_{-1,-1}$ $\quad$ $M_{1,1}$ $\quad$ $M_{-1,-1}$

trivial $\quad$ sign

2-dim $\quad M_z \quad z \in \mathbb{C}$

$$s \mapsto \begin{pmatrix} 1 & z \\ 0 & -1 \end{pmatrix} \quad t \mapsto \begin{pmatrix} -1 & 0 \\ \bar{z} & 1 \end{pmatrix}$$

$e \equiv 0 \bmod 2$	$e \equiv 1 \bmod 2$
M_z, z root (non-zero) of CP-poly	M_z, z root of CP-poly
$+ \mathbb{Z}/2\mathbb{Z}$ orbit	

Theorem: These are all.

Proof: Check that they are simple (we are talking about 2×2 matrices ☺)

Then: Sum squares $\rightsquigarrow \dim \mathbb{C}[W]$ ②

Example: $e = 4$ $\quad$ $\mathbb{Z}/2\mathbb{Z}$

$U_5 \rightsquigarrow$ roots $\{2\cos(\frac{\pi}{6}), \ldots, 2\cos(\frac{5\pi}{6})\}$

$+ 2\cos(\frac{3\pi}{6}) = 0$

$\rightsquigarrow$ 2 2-dim

$M_{2\cos(\frac{\pi}{6})} + M_{2\cos(\frac{5\pi}{6})}$

$\rightsquigarrow \dim(W) = 12 = 1^2 + 1^2 + 1^2 + 1^2$
$+ 2^2 + 2^2$ ✓

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(2) $\mathbb{Z}_+$-reps $\rightsquigarrow$ choice super important

$(P, B) \rightsquigarrow$ Algebra + basis such that

$$\boxed{xy \in \mathbb{N}B \qquad x, y \in B}$$

"Positive structure constants"

Similarly $\mathbb{Z}_+$-module (M, B^M)

$$\boxed{xm \in \mathbb{N}B^M}$$

Equivalence: $\mathbb{N}$-valued change of basis matrix

Examples: • Decategorification

• $\mathbb{C}[G]$ with group basis

Regular module is $\mathbb{Z}_+$

• Fusion rings, e.g.

$K_0(\mathrm{Rep}(G))$ "easy $\mathbb{Z}_+$-rep theory"

$K_0(\mathrm{Rep}_q^{ss}(\mathfrak{g}))$ "intricate $\mathbb{Z}_+$-RT"

• Hecke algebras for finite Coxeter groups with KL basis

Example: $\mathbb{C}[W] \rightsquigarrow$ KL basis

$\theta_1 = 1$, $\theta_s = 1+s$, $\theta_t = 1+t$

$\theta_{st} = 1+s+t+st$, $\theta_{ts} = 1+s+t+ts$

$\mathbb{Z}_+$-reps model ones ?

Cells: $x \leq_L y \iff \exists z$:

preorder

$\in B$

$\boxed{\text{$zx$ contains y with non-zero coefficient (in terms of B)}}$

$x \sim_L y \iff x \leq_L y \,,\, y \leq_L x$

equivalence relation

left cells

Similarly $x \leq_R y \,,\, x \leq_{LR} y$

$x \sim_R y \qquad x \sim_{LR} y$

right cells

two-sided cells

Similarly for (M, B^M)

Examples: • 1 is always in the lowest cell because

$$x \cdot 1 = x \Rightarrow 1 \leq_L x$$

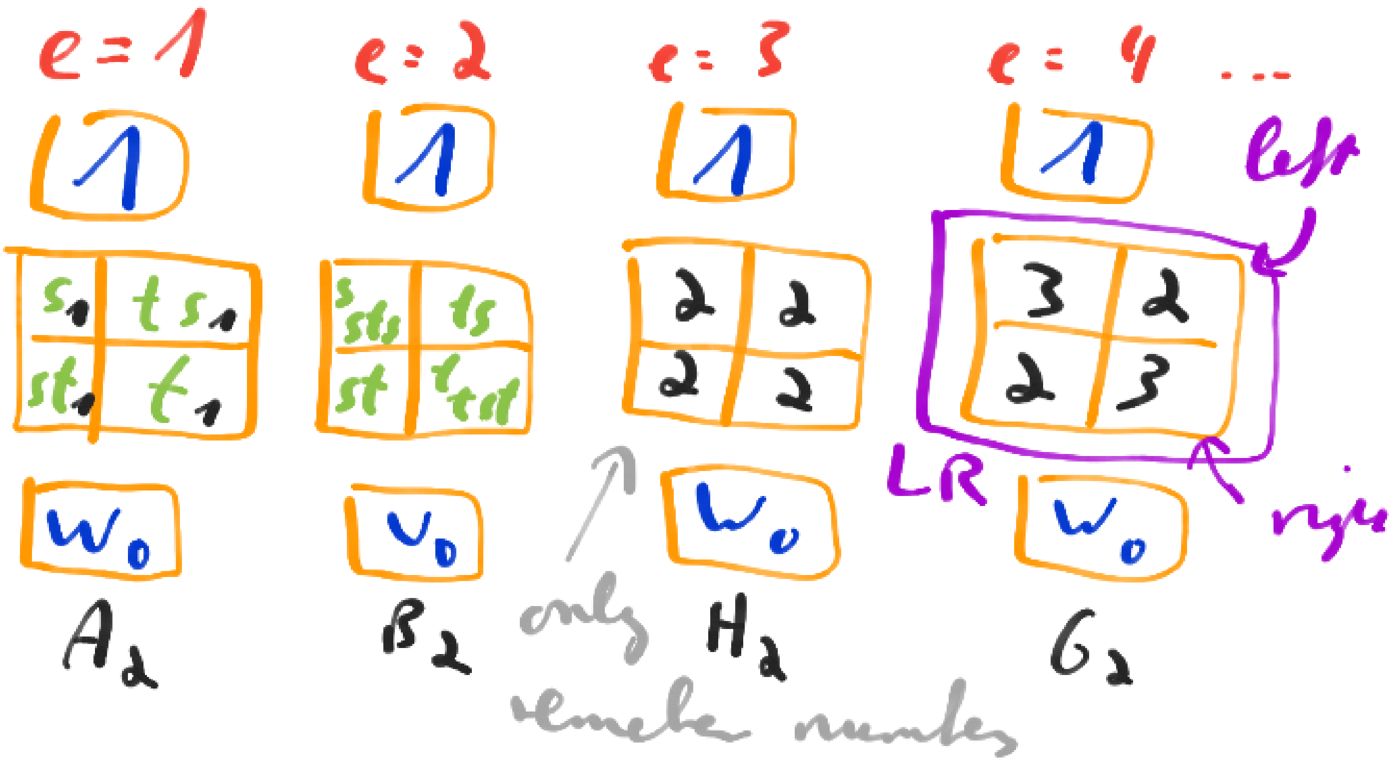
• $\mathbb{C}[G]$ has only one cell:

$$g^{-1}\cdot g = 1 \Rightarrow g \leq_L 1$$

• More generally Fusion alg. have only one cell

• Hecke algebras have very complicated cells

$$\boxed{\text{"Slogan: cells measure non-semisimplicity"}}$$



What about $\mathbb{Z}$-reps?

What are the atoms?

Transitive: One connected component under $\sim_L$ (left modules), $\sim_R$ (right modules) $\sim_{LR}$ (bimodules)

Fact: Each transitive module has a unique Apex, i.e. a max. LR-cell not killing it.

Examples: • $\mathbb{C}[G]$, apex $= G$, and transitive modules are $\mathbb{C}[G/H]$ for $H \subset G$ subgroup.

Why? Because each $g \in G$ has to act by an inv. (over $\mathbb{Z}_+$!) $\mathbb{Z}_+$-matrix and these are not so many of these

- Fusion algebras this is in gen. complicated, but:

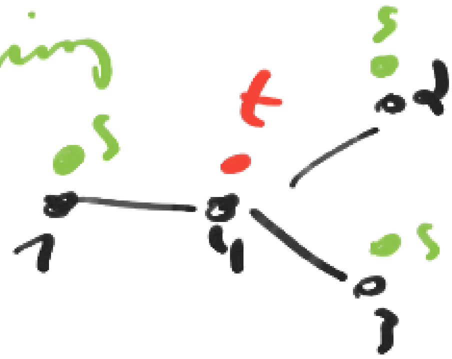
$K_0(\mathrm{Rep}(G)) \rightsquigarrow$ indexed by $H \subset G$

$K_0(\mathrm{Rep}_q^{ss}(sl_2)) \rightsquigarrow$ indexed by ADE

- Hecke algebras $\rightsquigarrow$ widely open

Dihedral example

+ coloring



$$A = \left(\begin{array}{ccc|c} & 0 & & 1 \\ & & & 1 \\ & & & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}\right)$$

$$s \rightsquigarrow \left(\begin{array}{ccc|c} & 1 & & 1 \\ & & & 1 \\ & & & 1 \\ \hline & & & -1 \end{array}\right) \qquad t \rightsquigarrow \left(\begin{array}{ccc|c} & -1 & & \\ & & & \\ \hline 1 & 1 & 1 & 1 \end{array}\right)$$

$$\underset{1+s}{\Theta_s} \rightsquigarrow \left(\begin{array}{ccc|c} & 2 & & 1 \\ & & & 1 \\ & & & 1 \\ \hline & & & 0 \end{array}\right) \qquad \underset{1+t}{\Theta_t} \rightsquigarrow \left(\begin{array}{ccc|c} & & & \\ \hline 1 & 1 & 1 & 2 \end{array}\right)$$

A, 0

$e+1$

$\rightsquigarrow M_\Gamma$

Lemma: M_Γ is a $\mathbb{Z}$-rep ← no $+$!
if and only if

$\Gamma = ADE$, $e+1 =$ Cox number

Proof: Use Smith ("we basically have replaced z by a $\mathbb{Z}_+$-matrix with z as eigenvalue") Not hard

Lemma: All transitive $\mathbb{Z}_+$-mods are of this form. no $+$!

Proof: A bit trickier, but basically still Smith

for the middle cell

Theorem: M_Γ give a complete list of transitive $\mathbb{Z}_+$-modules

Proof: Categorification

Apex	1	$\begin{array}{\|c\|c\|}\hline 3 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$	w_0
$\mathbb{Z}_0$-rep	$M_{-1,-1}$	M_{ADE}	$M_{1,1}$

Example $e = 2$

$\rightsquigarrow$ three notions of "atoms"

classical reps ; atoms = simples

	$M_{-1,-1}$	$M_{1,-1}$	$M_{\sqrt{2}}$	$M_{-1,1}$	$M_{1,1}$
name	sign		rotation		trivial
rank	1	1	2	1	1
Apex (KL)	1	big	big	big	w_0

non-trivial (Lusztig)
but simples have an apex as well !

Group el. bases, atoms $\xleftrightarrow{1:1}$ subgroups

H	1	$\langle st \rangle$	$\langle w_0 \rangle$	$\langle w_0, s \rangle$	$\langle w_0, st_s \rangle$	G
atom	reg					triv
rank	8	2	4	2	2	1
apex	G	G	G	G	G	G

$M_{11} \oplus M_{-1-1}$

$M_{\sqrt{2}} \oplus M_{\sqrt{2}}$

$M_{1,-1} \oplus M_{1,1}$

$M_{-1,1} \oplus M_{11}$

KL basis $\rightsquigarrow$ ADE + bicoloring

M_{-1-1}

$M_{1,1}$

sign				trivial
rank	1	3	3	1
apex	1	big	big	w_0

$M_{\sqrt{2}} \oplus M_{1,-1}$

$M_{\sqrt{2}} \oplus M_{-1,1}$