2-Verma modules

Grégoire Naisse
Joint work with Pedro Vaz

Université catholique de Louvain

22 November 2017

Highest weight representations

$\mathfrak{g}$ is a (symmetrizable) quantum Kac–Moody algebra.

There are 3 "kinds of" highest weight modules :

Highest weight representations

$\mathfrak{g}$ is a (symmetrizable) quantum Kac–Moody algebra.

There are 3 "kinds of" highest weight modules :

- finite-dimensional $V(\beta)$ (where β is integral),
  $\rightarrow F_i$ acts as nilpotent operator on any $w \in V(\beta)$;

Highest weight representations

$\mathfrak{g}$ is a (symmetrizable) quantum Kac–Moody algebra.

There are 3 “kinds of” highest weight modules :

- finite-dimensional $V(\beta)$ (where β is integral),
  $\rightarrow F_i$ acts as nilpotent operator on any $w \in V(\beta)$;
- Verma modules,
  $\rightarrow F_j^n w \neq F_j^m w$ for $w \neq 0$ and $n \neq m$;

Highest weight representations

$\mathfrak{g}$ is a (symmetrizable) quantum Kac–Moody algebra.

There are 3 "kinds of" highest weight modules :

- finite-dimensional $V(\beta)$ (where β is integral),
  $\to F_i$ acts as nilpotent operator on any $w \in V(\beta)$;
- Verma modules,
  $\to F_j^n w \neq F_j^m w$ for $w \neq 0$ and $n \neq m$;
- parabolic Verma modules,
  $\to$ a mix in-between (F_i locally nilpotent for some i's and 'infinite' for others).

(+tensor products...)

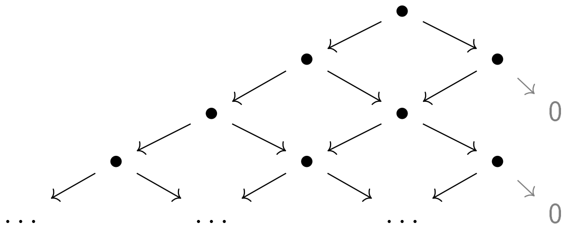
Highest weight representations : the picture

- $\mathfrak{p} \subset \mathfrak{g}$ is a (standard) parabolic subalgebra ;
- $V(\beta)$ $U_q(\mathfrak{p})$-module of highest weight β ;

Highest weight module

$$M^{\mathfrak{p}}(\beta) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V(\beta).$$

E.g. $\mathfrak{g} = \mathfrak{sl}_3 = \langle E_1, F_1, E_2, F_2, K_\gamma \rangle$ and $\mathfrak{p} = \langle E_1, E_2, F_2, K_\gamma \rangle$, $\beta = (\beta_1, 1)$.

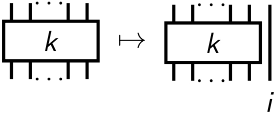


$$\Rightarrow M^{\mathfrak{p}}(\beta) \overset{\mathbb{Q}\text{-mod}}{\hookrightarrow} U_q^-(\mathfrak{g}) = \langle F_i's \rangle.$$

Categorification of $U_q^-(\mathfrak{g})$

KLR-algebras = braid-like algebras $R(k)$ with k-strands labeled by simple roots (+dots+relations)

$R(k) \hookrightarrow R(k+1)$:



$\mathsf{F}_i : R(k)\text{-mod} \to R(k+1)\text{-mod}$ = induction

Theorem (Khovanov–Lauda)

$$K_0\left(\bigoplus_k R(k)\text{-mod}\right) \cong U_q^-(\mathfrak{g})$$

as modules ober $U_q^-(\mathfrak{g})$ (and even more...)

$\Rightarrow$ KLR-algebras = good start to categorify H.W.M.

Some hints

$\tau : U_q(\mathfrak{g}) \to U_q^{op}(\mathfrak{g})$ anti-automorphism s.t. $\tau(F_i) \sim E_i$

Some hints

$\tau : U_q(\mathfrak{g}) \to U_q^{op}(\mathfrak{g})$ anti-automorphism s.t. $\tau(F_i) \sim E_i$

$\Rightarrow$ it defines a sequilinear form $\langle -, - \rangle$ on $M^{\mathfrak{p}}(\beta)$ s.t.

$$\langle F_i -, - \rangle \sim \langle -, E_i - \rangle.$$

Some hints

$\tau : U_q(\mathfrak{g}) \to U_q^{op}(\mathfrak{g})$ anti-automorphism s.t. $\tau(F_i) \sim E_i$

$\Rightarrow$ it defines a sequilinear form $\langle -, - \rangle$ on $M^{\mathfrak{p}}(\beta)$ s.t.

$$\langle F_i -, - \rangle \sim \langle -, E_i - \rangle.$$

$\Rightarrow$ it looks like gdim (=decategorification) of some HOM, and

$$\mathrm{HOM}(\mathrm{F}_i -, -) \sim \mathrm{HOM}(-, \mathrm{E}_i -).$$

Some hints

$\tau : U_q(\mathfrak{g}) \to U_q^{op}(\mathfrak{g})$ anti-automorphism s.t. $\tau(F_i) \sim E_i$

$\Rightarrow$ it defines a sequilinear form $\langle -, - \rangle$ on $M^{\mathfrak{p}}(\beta)$ s.t.

$$\langle F_i -, - \rangle \sim \langle -, E_i - \rangle.$$

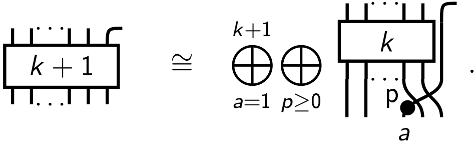
$\Rightarrow$ it looks like gdim (=decategorification) of some HOM, and

$$\mathrm{HOM}(\mathsf{F}_i -, -) \sim \mathrm{HOM}(-, \mathsf{E}_i -).$$

$\Rightarrow$ E_i "should be" right adjoint of F_i, hence restriction functor.

Restriction functor

We get a decomposition



$\Rightarrow$ basically E_i on $R(k+1)$-mod acts “multiplication” by

$$q^{k_1}\frac{1}{1-q^2}+\cdots+q^{k_\ell}\frac{1}{1-q^2}$$

where $\frac{1}{1-q^2}=1+q^2+q^4+\ldots$

$\Rightarrow$ we need to modify the KLR-algebras.

Finite case

If F_i is locally nilpotent (=> H.W. integral), E_i should acts as

$$q^{k_1}[m_1] + \cdots + q^{k_\ell}[m_\ell],$$

where $[m_s] = q^{m_s - 1} + \cdots + q^{1-m_s}$.

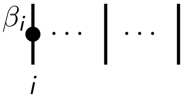
Finite case

If F_i is locally nilpotent (=> H.W. integral), E_i should acts as

$$q^{k_1}[m_1] + \cdots + q^{k_\ell}[m_\ell],$$

where $[m_s] = q^{m_s - 1} + \cdots + q^{1-m_s}$.

$\Rightarrow$ KLR is too big $\rightarrow$ we take quotient : the cyclotomic quotient



$= 0.$

Infinite/Verma case

For F_i 'infinite', E_i should acts as

$$q^{k_1}\frac{q^{m_1}\lambda_i - q^{-m_1}\lambda_i^{-1}}{1-q^2} + \cdots + q^{k_\ell}\frac{q^{m_1}\lambda_i - q^{-m_1}\lambda_i^{-1}}{1-q^2},$$

where $\lambda_i = q^{\beta_i}$ (=formal parameter).

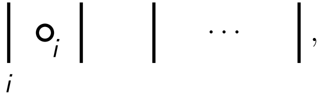
Infinite/Verma case

For F_i 'infinite', E_i should acts as

$$q^{k_1}\frac{q^{m_1}\lambda_i - q^{-m_1}\lambda_i^{-1}}{1-q^2} + \cdots + q^{k_\ell}\frac{q^{m_1}\lambda_i - q^{-m_1}\lambda_i^{-1}}{1-q^2},$$

where $\lambda_i = q^{\beta_i}$ (=formal parameter).

$\Rightarrow$ KLR is too small $\rightarrow$ add grading λ_i and superstructure (parity), with new generator



of q-degree $= 0$, λ_i-degree $= 2$ and parity $= 1$ (they anticommute).

Some facts

For F_i locally nilpotent, $\mathsf{E}_i, \mathsf{F}_i$ realize (categorical) $\mathfrak{sl}(2)$-commutator as direct sum, otherwise there is a natural SES :

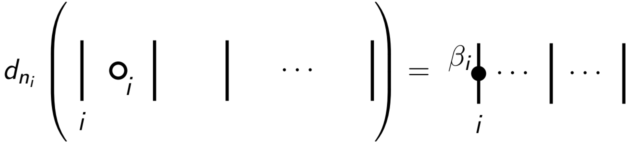
$$0 \to \mathsf{F}_i\mathsf{E}_i \to \mathsf{E}_i\mathsf{F}_i \to \frac{1}{q-q^{-1}}(K \oplus \Pi K^{-1}) \to 0.$$

Some facts

For F_i locally nilpotent, $\mathsf{E}_i, \mathsf{F}_i$ realize (categorical) $\mathfrak{sl}(2)$-commutator as direct sum, otherwise there is a natural SES :

$$0 \to \mathsf{F}_i\mathsf{E}_i \to \mathsf{E}_i\mathsf{F}_i \to \frac{1}{q - q^{-1}}(K \oplus \Pi K^{-1}) \to 0.$$

Also, there is a differential



so that the homology is a cyclotomic quotient.

$\Rightarrow$ we got a dg-enhanchement of cyclotomic-KLR,
$\Rightarrow$ it allows to compute many things “easily” in these.

A bit of topology

HOMFLY polynomial arises in parabolic Verma modules of $\mathfrak{gl}(2n)$ (by Queffelec-Sartori work),
$\Rightarrow$ KR-HOMFLY homology arises naturally in the parabolic 2-Verma modules,
$\Rightarrow$ we have new tools to compute stuff about KR homology.

A bit of topology

HOMFLY polynomial arises in parabolic Verma modules of $\mathfrak{gl}(2n)$ (by Queffelec-Sartori work),
$\Rightarrow$ KR-HOMFLY homology arises naturally in the parabolic 2-Verma modules,
$\Rightarrow$ we have new tools to compute stuff about KR homology.

Theorem (N., Vaz, 2017)

d_N induces a spectral sequence on KR-homology to $\mathfrak{sl}(N)$-homology, which agree with Rasmussen's one. Moreover, the SS converges at the second page.

$\rightarrow$ Idea : lift the complex in the 'total' dg-enhancement and observe it is homototopic to a complex concentrated in degree 0.