<u>Disclaimer</u>: all cats are small

<u>Motivation:</u>

Classical representation theory

$\Bbbk = \overline{\Bbbk}$ $\quad A =$ f.d. $\Bbbk$-algebra

$\rightsquigarrow$ A-mod: objects $V = \Bbbk$-vector space with A-action

This is really a simplification. Should really consider $(A\text{-proj})^{op}$ and consider additive functors to the category of (f.d.) $\Bbbk$-vector spaces

$\rightsquigarrow$ $(A\text{-proj})^{op}\text{-mod} \simeq A\text{-mod}$

Higher representation, or 2-representation theory replaces $(A\text{-proj})^{op}$ with a 2-category and $\Bbbk$-vector spaces with "canonical 2-category" and functors with 2-functors,

<u>2-categories</u>

<u>Defn</u> A **2-category** is a category enriched over CAT, the category of all (small) categories

This means that a 2-category is a category with
- objects
- each $\mathcal{C}(i,j)$ is a category

morphisms

- Composition is bifunctorial

$$\mathcal{C}(j,k)\times\mathcal{C}(i,j)\to\mathcal{C}(i,k)$$

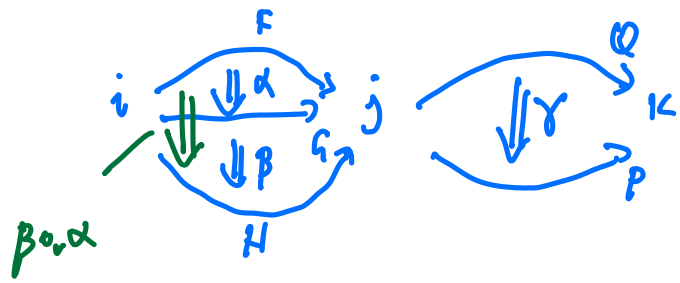
- identity objects $1_i \in \mathcal{C}(i,i)$ $\forall i$

+ satisfying natural (strict) axioms

Terminology

1) Objects in $\mathcal{C}(i,j)$ are called 1-morphisms
2) Morphisms in $\mathcal{C}(i,j)$ —— 2-morphisms
3) Composition in $\mathcal{C}(i,j)$ — $\circ_v$ = vertical comp.
4) —— $\mathcal{C}$ — $\circ_H$ = horizontal

Structural unit in $\mathcal{C}$



$\gamma\circ\alpha : Q\circ F \to P\circ G$

Examples

① CAT : objects: categories

1-morphisms: functors

2-morphisms: natural transformations

1_A : ID_A A- category

composition = composition

② $(S, \circ, e, \leq)$ ordered monoid

multiplication, identity, partial order compatible with left and right multiplication

$\mathcal{C} = \mathcal{C}(S, \circ, e, \leq)$

one object : $\#$

1-morphisms : $\mathcal{C}(\#, \#) = S$

$\mathbb{1}_\# = e, \quad \circ_\# = \circ$

2-morphisms :

$$\mathrm{Hom}(s,t) = \begin{cases} \phi, & \text{if } s \not\leq t \\ \{h_{st}\}, & \text{if } s \leq t \end{cases} \qquad s,t \in S$$

formal element.

$\Rightarrow \circ_v$ is uniquely defined.

③ A-mod-A, for a f.d. k-algebra

Objects : $\#$

1-morphisms : f.d. (A,A)-bimodules

2-morphisms : bimodule homs

$$\circ_H = \otimes_A$$
$$\circ_v = \text{composition}$$
$$\mathbb{1}_{\#} \simeq {}_A A_A$$

This only enough to define a "bicategory"
$\rightsquigarrow$ strictification leads to a 2-category.

2-Functors

A 2-functor is a functor between 2-cats that preserve all additional structure

$$\begin{tikzcd} i \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & j \end{tikzcd} \xrightarrow{\ \Phi\ } \begin{tikzcd} \Phi(i) \arrow[r, bend left, "\Phi(F)"] \arrow[r, bend right, "\Phi(G)"'] \arrow[r, phantom, "\Downarrow \Phi(\alpha)"] & \Phi(j) \end{tikzcd}$$

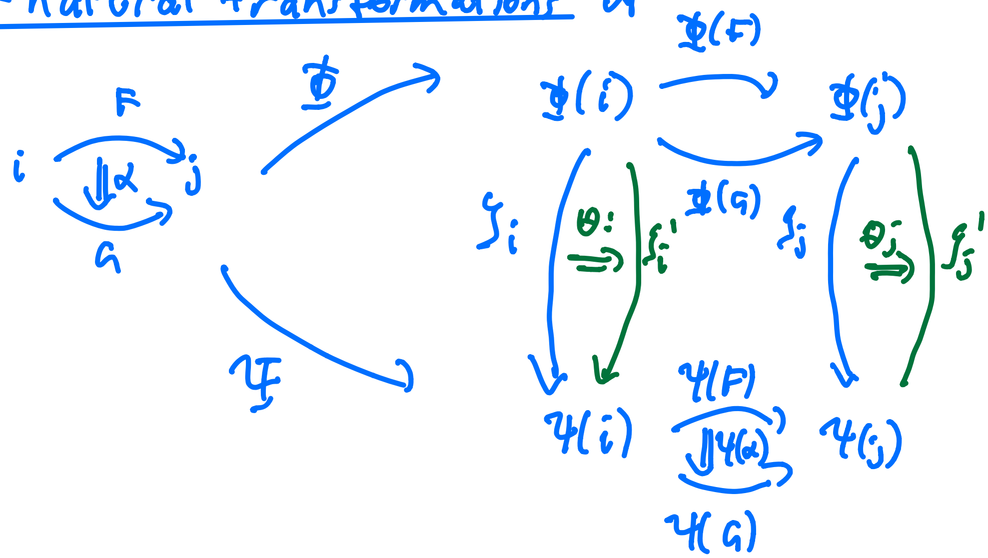
$$\mathcal{C} \longrightarrow \mathcal{C}'$$

+ respect $\circ_H$, $\circ_v$, $\mathbb{1}_i$ and id_F

Example $\mathcal{C}$ = 2-cat $\quad i \in \mathcal{C}$

$$\mathcal{C}(i, \_) : \mathcal{C} \to CAT$$
$$j \mapsto \mathcal{C}(i,j) \in CAT$$
$$j \xrightarrow{F} k \longmapsto F \circ \_ : \mathcal{C}(i,j) \to \mathcal{C}(i,k)$$
$$\alpha : F \to G \longmapsto \alpha \circ \_ : F \circ \_ \to G \circ \_$$

2-natural transformations $\mathcal{A}$



A 2-natural transformation $\rho : \Phi \Rightarrow \Psi$ is a map $\mathrm{Ob}(\mathcal{C}) \to 1\text{-}\mathrm{Mor}(\mathcal{A})$ such that the diagram above commutes

<u>Modifications</u> $\rho \xrightarrow{\Theta} \rho'$ such that the diagram above commutes

$\Rightarrow$ 2-CAT is a 3-CAT (but lets not get into this...)

Let us fix 2-categories $\mathcal{C}$ and $\mathcal{A}$

2-CAT$(\mathcal{C}, \mathcal{A})$: objects = 2-functors
1-morphisms = 2-natural trans,
2-morphisms = modifications

This is the category of representations of $\mathcal{C}$ in $\mathcal{A}$

For these lectures choose $\mathcal{A}$ to be the 2-category of finitary "$\Bbbk$-linear cats"

A <u>finitary</u> $\Bbbk$-linear category is a category equivalent to A-proj for some f.d. $\Bbbk$-algebra A.

1-morphisms: additive $\Bbbk$-linear functors

2-morphisms: natural transformations

<u>Remark</u> Sometimes one needs a weaker notion of 2-natural transformations
— see: Leinster, "Basic bicategories"

<u>Alternatively</u> "module theoretic perspective"
A 2-rep. of $\mathcal{C}$ in $\mathcal{A}$ is a functorial action of $\mathcal{C}$ on categories in $\mathcal{A}$ (if $\mathcal{A}$ is a subcategory of Cat)

<u>Finitary 2-categories</u>
These are natural 2-analogues of f.d. algebras via $A\text{-mod} \simeq (A\text{-proj})^{op}\text{-mod}$
take a subcategory of indec. objects

A 2-category $\mathcal{C}$ is finitary over $\Bbbk$ if
1) $\mathcal{C}$ has finitely many objects
2) Each $\mathcal{C}(i,j) \simeq A_{ij}\text{-proj}$, for some f.d. $\Bbbk$-algebra A_{ij}

3) All compositions are biadditive and $\Bbbk$-bilinear (whenever applicable)

4) All $\mathbb{1}_i$ are indecomposable

<u>Examples</u>

1) $\mathcal{C}_A$, A a f.d. $\Bbbk$-algebra (with connected Gabriel quiver)

1-object: think of it as A-mod

1-morphisms: endofunctors of A-mod in $\mathrm{add}(A, \underbrace{A \otimes_{\Bbbk} A}_{\text{projective endofunctors}})$

2-morphisms: natural transformations

$\mathcal{C}_A$ = projective endofunctors of A-mod

2) A-mod-A

object $\sim$ A-mod

This 2-category is finitary if and only if it is a quotient of

$$\circ \to \circ \to \circ \to \circ \cdots \circ \to \circ \,/\, \mathrm{Rad}^2 = 0$$

<u>Lecture II</u>

$\mathcal{C}$ a finitary 2-category

Want to study category of additive 2-reps of $\mathcal{C}$

2-cat $(\mathcal{C}, \mathcal{A})$ ← finitary k-linear categories

i.e. functorial actions of $\mathcal{C}$ on a category of the form B-proj, by additive k-linear functors

Question What is the correct notion of "simple" 2-reps?

Defn A 2-rep. M is transitive if for any indecomposable objects $X \in M(i)$ and $Y \in M(j)$ there exists $F \in \mathcal{C}(i,j)$ such that Y is isomorphic to a summand of $F(X)$ $\left(= M(F)(X)\right)$

This places restrictions only on objects, not morphisms.

Defn M is simple if $\coprod_i M(i)$ has no proper $\mathcal{C}$-invariant ideal

Remark Simple ⇒ Transitive

So "simple transitive 2-reps" are "proper" analogues of simple modules over algebras.

Eg. There is a weak analogue of Jordan-Hölder theory for additive 2-reps.

Problem Classify all simple transitive 2-reps of a given 2-category $\mathcal{C}$.

Open problem For a given $\mathcal{C}$, is the number of simple transitive 2-reps finite?

$\longleftrightarrow$ for a f.d. algebra A, the number of isomorphisms classes of simple A-modules is finite

<u>Remark</u> Chuang-Rouquier allow you to classify simple transitive 2-reps for finitary quotients of $U(\mathfrak{sl}_2)$

This is somehow "easy" because there are lots of idempotents $1_{\underline{i}}$.

<u>Ex</u> A a f.d. $\Bbbk$-algebra

$\rightsquigarrow \mathscr{C}_A$ = 2-cat of projective (A, A)-bimodules

The defining 2-rep gives a functorial action of $\mathscr{C}_A$ on A-proj.

<u>Lemma</u> This 2-rep is simple transitive

<u>Proof</u> transitive: $P_1 = Ae_1, \dots, P_k = Ae_k$ proj. indecomposable A-modules

$\rightsquigarrow$ $Ae_j \otimes e_i A \otimes_A Ae_i$ has Ae_j as a summand

<u>Simple:</u> exercise $\quad$ //

<u>Ex</u> A a f.d. $\Bbbk$-algebra

$\mathscr{C}_A$ = 2-cat. of proj. (A,A)-bimodules (basic, connected)

$A \neq \Bbbk$, $\mathscr{C}_A$ acts on $\Bbbk$-mod

${}_A A_A$ acts on $\mathrm{Id}_{\Bbbk\text{-mod}}$

$A \otimes_{\Bbbk} A$ —— 0

__Theorem__ (M-Miemetz-Zhang '17)

These are the only simple transitive 2-reps of $\mathscr{C}_A$, up to equivalence.

__Remark__ Various special cases proved previously

__Remark__ $A = \Bbbk$, $\mathscr{C}_{\Bbbk}$ acts on B-proj for any B

Simple transitive $\Leftrightarrow$ $B = \Bbbk$ $\simeq \mathscr{C}_{\Bbbk}(\#, \_)$

__Example__ $D = \Bbbk[x]/(x^2)$

$\mathscr{C}_D$ $\quad \mathbb{1} = {}_D D_D$, $\quad F = D \otimes_{\Bbbk} D$

$\circ$	$\mathbb{1}$	F
$\mathbb{1}$	$\underline{\mathbb{1}}$	F
F	F	$F \oplus F$

$$F \circ F = D \otimes_{\Bbbk} D \otimes_D D \otimes_{\Bbbk} D$$
$$\simeq (D \otimes_{\Bbbk} D)^{\oplus 2}$$

$\mathscr{C}_D \simeq$ 2-category of Soergel bimodules of type A over the coinvariant algebra

Let M be a simple transitive 2-rep of $\mathscr{C}_D$

$M(\#) = B$-proj $\qquad P_1 \_ P_n$ indec. proj. B-mods

Define a matrix $[F] = (m_{ij})_{i,j=1}^{n} =: Q$
where m_{ij} = multiplicity of P_i in $F(P_j)$ (*)

Now, $F^2 = F \oplus F \Rightarrow Q^2 = 2Q$

M transitive $\Rightarrow m_{ij} \neq 0$ if $i \neq j$

(*) $\Rightarrow m_{ij} \neq 0$ for all $i,j \in I$

$\Rightarrow Q \in Mat_n(\mathbb{Z}_{>0})$, so by Frobenius–Perron,
$\exists!$ eigenvalue λ of maximal $|\ \ |$, which is simple

As $Q^2 = 2Q \Rightarrow \lambda = 2$ and all other eigenvalues are equal to 0

$\Rightarrow$ Q has rank 1

$\Rightarrow$ $Q = (2)$ or $Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow$ $n \leq 2$.

D is self-injective $\Rightarrow \mathcal{C}_D$ is a fiat 2-category

<u>Defn</u> A 2-category $\mathcal{C}$ is fiat if it has a weak involution $*: \mathcal{C} \to \mathcal{C}$ (anti-auto. reversing 1 and 2-morphisms
such that (F, F^*) is an adjoint pair of 1-morphisms

<u>Lemma</u> If A is self-injective, weakly symmetric then $\mathcal{C}_A$ is fiat

Pf $\mathrm{Hom}_{A-}(Ae \otimes_{\Bbbk} fA, M)$

$$\simeq \mathrm{Hom}_{\Bbbk}(fA, \mathrm{Hom}_A(Ae, M))$$

$$\simeq \mathrm{Hom}_{\Bbbk}(fA, \Bbbk) \otimes_{\Bbbk} eA \otimes_A M$$

$\Rightarrow M(F)$ is an exact functor

Q. How do we construct A-mod from A-proj?

$$A\text{-proj} \overset{\text{abelianisation}}{\rightsquigarrow} A\text{-mod} = \overline{A\text{-proj}}$$

Lemma Assume M is a simple 2-rep
Then $\overline{M}(F)$ sends simples to projectives

$[F] = 2 \Rightarrow \dim B = 2$ $\qquad [F] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow B$ is the defining 2-rep. $\qquad \Rightarrow B = \Bbbk \oplus \Bbbk$

$\Rightarrow D = \mathrm{End}({}_A A_A) \hookrightarrow \Bbbk \oplus \Bbbk$

$\rightarrow$ ※

Lecture III

Cells and cell 2-representations

$\mathscr{C}$ a finitary 2-category

Defn F, G 1-morphisms in $\mathscr{C}$

Then $F \geq_L G$ if $\exists H$ s.t. $F \simeq$ a summand HG

$F \geq_R G$ if $\exists H$ s.t. $F \simeq$ a summand GH

$F \geq_J G$ if $\exists\, H_1, H_2$ s.t. $F \simeq$ a summand $H_1 G H_2$

These are the left, right and two-sided preorders

$\rightsquigarrow$ cells = corresponding equiv. classes

<u>Example</u> $\mathcal{C}_A$, A basic, $A \neq \Bbbk$, connected

$1 = e_1 + \dots + e_n$ pairwise orth. primitive idempotents

Indec 1-morphism in $\mathcal{C}_A$: $\mathbb{1} \simeq {}_A A_A$

$$F_{ij} \simeq Ae_i \otimes e_j A$$

$\mathbb{1}$ = left, right, two-sided cells

$$\rightsquigarrow \quad \begin{matrix} F_{11} & F_{12} & \dots & F_{1n} \\ | & & & \\ F_{n1} & F_{n2} & \dots & F_{nn} \end{matrix}$$

right cells

left cells

<u>Remark</u> If $\mathcal{L}$ is a left cell $\Rightarrow \exists\, i \in \mathcal{C}$ s.t. all 1-morphisms in $\mathcal{L}$ start at i. Define $i_{\mathcal{L}} = i$

Consider $\mathcal{C}(i, -)$ - Yoneda 2-rep.

Let $M_{\mathcal{L}}$ be the 2-rep. given by $\mathcal{C}$ acting on $\operatorname{add}(F,\ F \geq_L \mathcal{L})$

$\coprod_j \mathcal{C}(i,j)$

<u>Lemma</u> $M_{\mathcal{L}}$ has a unique maximal $\mathcal{C}$-stable ideal $I_{\mathcal{L}}$

<u>Defn</u> The cell 2-rep $C_{\mathcal{L}} := M_{\mathcal{L}} / I_{\mathcal{L}}$

By the lemma, $C_{\mathcal{L}}$ is a simple transitive 2-rep

<u>Remark</u> Indecomposable objects in $\coprod_j C_{\mathcal{L}}(j)$ are in bijection with 1-morphisms in $\mathcal{L}$

<u>Examples</u>

- $\mathcal{C}_A$, $\mathcal{L} = \{ \mathbb{1} \} \Rightarrow M_{\mathcal{L}} = \mathcal{C}(\#, -)$

  $I_{\mathcal{L}}$ contains id_F, Rad End$(\mathbb{1})$

  $\Rightarrow$ underlying category of $C_{\mathcal{L}}$ is $\Bbbk$-mod

- $\mathcal{L}_2 = \{ F_{i,1} \}$. Then $M_{\mathcal{L}_i} = \text{add}(F_{i1})$

  $\mathcal{C}$ acts on $\text{End}(A \otimes_{\Bbbk} e_1 A)^{op}$-proj

  $I_{\mathcal{L}_2} \supseteq \text{End}(A \otimes_{\Bbbk} \text{rad}\, e_1 A)^{op}$

  $\Rightarrow$ under lying category of $C_{\mathcal{L}_2}$ is A-proj,

the defining 2-rep. of $\mathcal{C}_A$

Last time we saw that these are the

only two possible simple tran. 2-reps.

$\Rightarrow \mathcal{L}' = \{F_{ij}\} \qquad C_{\mathcal{L}'} \simeq C_{\mathcal{L}_2}$

In this case, simple transitive $\Leftrightarrow$ cell 2-rep

<u>Soergel bimodules</u>

Let (W,S) be a finite Coxeter system

Let V be a reflection faithful rep. of W

$\rightsquigarrow$ $C = C(W,S,V) =$ coinvariant algebra

$\Rightarrow$ $\dim C = |W|$

The 2-category $\mathcal{S}$ of Soergel bimodules

has 1 object $\#$ (think of as C-mod)

1-morphisms: endofunctors of C-mod

$\simeq \otimes$ with bimodules in $\mathrm{add}(C\text{-})$

For all $s \in S$, C^s = s-invariants in C

$\forall s_1, \ldots, s_k \qquad C \otimes_{C^{s_1}} \otimes \cdots \otimes_{C^{s_{k-1}}} C \otimes_{C^{s_k}} C$

2-morphisms: natural transformations

<u>Q</u>. Is $\mathcal{S}$ finitary?

Theorem (Soergel) If $w \in W$, $w = s_1 \cdots s_k$ reduced, then $C \otimes_{C^{s_1}} \otimes \cdots \otimes_{C^{s_k}} C$ contains a unique indecomposable summand that is not in $\operatorname{add}(\theta_x, \ell(x) < \ell(w))$ and $\{\theta_w \mid w \in W\}$ is a complete set of indec. Soergel bimodules

Corollary $\mathscr{S}$ is finitary

Theorem (Soergel, Elias-Williamson)

$$K_0(\mathscr{S}) \simeq \mathbb{Z}[W]$$

$$[\theta_s] \longmapsto \underline{H}_w = \text{Kazhdan-Lusztig basis element}$$

Remark type $A_1 \Rightarrow C = D = \mathbb{C}[x]/(x^2)$
$\Rightarrow \mathscr{S} \simeq \mathscr{C}_D$

Open problem Classify simple transitive 2-reps of $\mathscr{S}$

Type A cells = KL-cells (all types)

When $W = S_n$, the KL-cells are particular nice. By Robinson-Schensted,

$$W \simeq \coprod_{\lambda \vdash n} \mathrm{Std}(\lambda)^2 ; \; w \mapsto (P_w, Q_w)$$

Thm (Kazhdan-Lusztig)

If $x, y \in S_n$ then

$$x \sim_L y \iff P_x = P_y$$

$$x \sim_R y \iff Q_x = Q_y$$

$$x \sim_J y \iff \mathrm{shape}(P_x) = \mathrm{shape}(P_y)$$

Corollary Let J be a two-sided cell, L a left cell in J, and R a right cell in J. Then $|L \cap R| = 1$.

Defn A two-sided cell J is regular if $|L \cap R| = 1$ whenever L is a left cell in J and R is a right cell in J.

Theorem (M.-Miemietz)

Suppose that $\mathcal{C}$ is fiat and that all two-sided cells in $\mathcal{G}$ are regular. Then

① simple transitive $\Rightarrow$ cell 2-rep

② $C_h \simeq C_{h'} \iff h \sim_J h'$

<u>Corollary</u> In type A, simple transitive $\equiv$ cell 2-reps.

<u>Dihedral groups</u>

$n \geq 3$, $I_2(n)$ $S = \{s,t\}$ $w_0 = \underbrace{sts\ldots}_{n} = \underbrace{tst\ldots}_{n}$

Bruhat

$$\begin{tikzcd}[column sep=small] & e \arrow[dl, dash] \arrow[dr, dash] & \\ s \arrow[d, dash] \arrow[drr, dash] & & t \arrow[d, dash] \arrow[dll, dash] \\ st & & ts \\ & \vdots & \\ & w_0 & \end{tikzcd}$$

Key fact: $\underline{H}_w = \sum_{x \leq w} H_x$

<u>Cells</u>

$\{e\}$

$\{s \ldots s\}$ $\{s \ldots t\}$

$\{t \ldots s\}$ $\{t \ldots t\}$ $\Big\}$ big cells

$\{w_0\}$

$|W| = 2n$	n odd	n even
Size of "big" cell:	$\frac{n-1}{2}$	$\frac{n}{2}$

Let $\underline{S} = S/(\theta_{w_0})$

$n=3 \Rightarrow W = S_3 \Rightarrow$ simple trans. $\approx$ cell 2-rep

<u>$n=4$</u> <u>Theorem</u> (Zimmerman)

1) simple transitive $\Rightarrow$ cell
2) $C_{h_s} \not\simeq C_{h_t}$

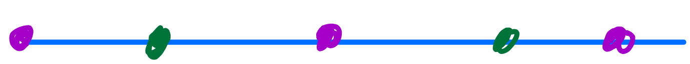
<u>n odd</u> <u>Theorem</u> (Kildetoft–Mackaay–M–Zimmerman)

1) simple trans $\Rightarrow$ cell
2) $C_{h_s} \simeq C_{h_t}$

<u>n even</u> underlying algebra of C_{h_s}

s •
t •

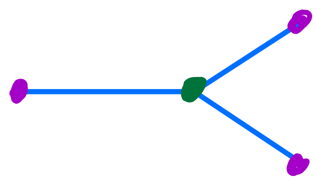
$s \rightleftarrows ts \rightleftarrows sts \rightleftarrows$



As n is even this graph as a symmetry

$\Rightarrow \exists\ \Theta \in \mathrm{Aut}(C_{h_s})$ s.t. $\Theta^2 = \mathrm{id}$

$\mathcal{S}'$ acts on $C^{\Theta}_{\mathcal{L}_s}$ "orbit construction"



<u>Theorem</u> (KMMZ) $n > 4$.

$\mathcal{S}'$ has 5 simple transitive 2-reps

$$C_{\mathbb{1}}, \quad C_{\mathcal{L}_s}, \quad C_{\mathcal{L}_t}, \quad C^{\Theta}_{\mathcal{L}_s} \quad C^{\Theta}_{\mathcal{L}_t}$$

and these are all unless $n = 12, 18, 30$

These are the Coxeter numbers for E_6, E_7, E_8

<u>Lecture IV</u>

(W, S) a Coxeter system of type $I_2(m)$, m even > 4.

$\mathcal{D}$ = "small" quotient of Soergel bimodules

Two-sided cells:

Θ_e

$\Theta_s \dots \Theta_s$	$\Theta_s \dots \Theta_t$	$\frac{n}{2}$	$\frac{n}{2}-1$
$\Theta_t \dots \Theta_s$	$\Theta_t \dots \Theta_t$	$\frac{n}{2}-1$	$\frac{n}{2}$

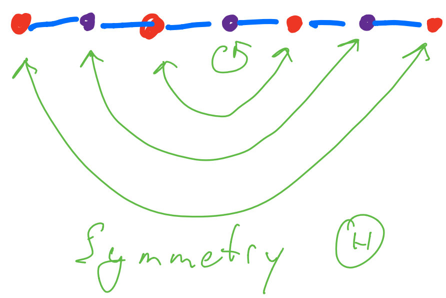
$C_{\mathcal{L}_s}$ = cell 2-rep. of $\mathcal{D} \Rightarrow$ acts on B-proj

$$s \rightleftarrows ts \rightleftarrows sts \rightleftarrows \cdots \qquad \begin{array}{l} \to\to = 0 \\ \leftarrow\leftarrow = 0 \\ \leftarrow\to = 0 \end{array}$$

C_{L_e}, C_{L_s}, C_{L_t}: $s = s$ $t = t$

Θ_s kills •

Θ_t kills •



Type A 2-rep

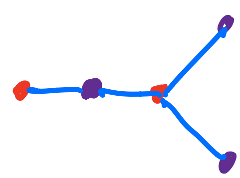
$\Theta \in \mathrm{End}(C_{L_s})$, $\Theta^2 = \mathrm{id}$

Orbit category

G acting on $\mathcal{A}$ $\rightsquigarrow$ $i \xrightarrow{\bigoplus_{g \in G} \mathrm{Hom}(i, g(j))} j$

$\Rightarrow$ more automorphisms

In our case:



$\mathcal{A}$ acts

$\Rightarrow$ type D 2-rep: $C^{\Theta}_{L_s}$, $C^{\Theta'}_{L_t}$

Theorem (Kildetoft–Mackaay–M–Zimmermann)

$\mathcal{A}$ has exactly five equivalence classes of simple transitive 2-representations:

$$C_{L_e},\ C_{L_s},\ C_{L_t},\ C^{\Theta}_{L_s}\ C^{\Theta}_{L_t}$$

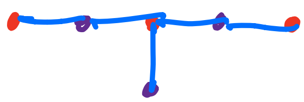
unless $n = \underbrace{12, 18 \text{ or } 30}_{\text{Coxeter numbers of } E_6, E_7, E_8}$.

<u>Theorem</u> (Mackaay–Tubbenhauer)

For $n = 12, 18$ and 30 there are two more equivalence classes of simple transitive 2-reps and, assuming gradability, there are no more.

<u>Open problem</u> Remove gradability assumption

<u>Example:</u> $n = 12$ the Coxeter number of E_6



$\mathcal{A}$ acts

<u>Proof:</u> Brute force check using Soergel bimodule relations

<u>Question</u> Is there a more conceptual proof

<u>Abelianisations</u>

$\mathscr{C}$ a finitary 2-category $\Rightarrow$ $\mathscr{C}$-proj abelianisation

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d] \arrow[dl, "b"'] \\ X' \arrow[r, "\alpha'"'] & Y' \end{tikzcd}$$

$\exists\ a = \alpha' b$ for some b

$\mathscr{C}$-injective abelianisation: mod out other triangle

$\mathcal{C}$-fiat : $\star : \mathcal{C} \to \mathcal{C}$
additive

$$\begin{tikzcd} & \text{abelian categories} \arrow[dl] \arrow[dr] & \\ \star \;\; \mathcal{C} \arrow[rr, shift left] & & \mathcal{C} \arrow[ll, shift left] \end{tikzcd}$$

⇝

Algebra objects (1-morphisms)

$\Bbbk$-algebras	$\mathcal{C}$ 2-category
multiplication	F 1-morphism
$A \otimes_{\Bbbk} A \xrightarrow{\text{mult.}} A$	$FF \xrightarrow{\mu} F$
$\Bbbk \xrightarrow{\text{unit}} A$	$\mathbb{1} \xrightarrow{\eta} F$

axioms

eg. associativity $\leftrightarrow$

$$\begin{tikzcd} FFF \arrow[r, "\mu \circ \mathrm{id}"] \arrow[d, "\mathrm{id} \circ \mu"'] & FF \arrow[d, "\mu"] \\ FF \arrow[r, "\mu"] & F \end{tikzcd}$$

Can do the same for left modules:

$A \otimes_{\Bbbk} M \longrightarrow M$ $FG \longrightarrow G$

+ axioms

Similarly for right modules, bimodules, comodules etc etc.

Application to finitary 2-categories

In the set up of fusion categories (~abelian bicategory with $\oplus$) is due to Ostrik – see Etinghof–Gelaki–Ostrik

Adaption to finitary 2-categories:

Mackaay – M. – Miemietz – Tubbenhauer

Uses: internal hom:

$\mathbf{M}$ finitary 2-rep of $\mathscr{C}$

$M \in \mathbf{M}(i)$, $N \in \mathbf{N}(j)$ have

$$\underset{\underset{\mathscr{G} = \mathrm{Inj}(\mathscr{C})}{\cap}}{F} \longrightarrow \mathrm{Hom}_{\mathbf{M}}(M, FN) \underset{\mathrm{Vect}}{\cap}$$

$\exists!$ ext. up to isomorphism to a left exact functor $\underline{\mathscr{C}} \longrightarrow \mathrm{Vect}$

Representability $\Rightarrow$ up to iso, $\exists!$

$$\underline{\mathrm{Hom}}(M, N) \in \mathscr{G}$$

such that

$$\mathrm{Hom}_{\mathbf{M}}(M, FN) = \mathrm{Hom}_{\mathscr{C}}(\underline{\mathrm{Hom}}(M, N), F)$$

for all $F \in \mathscr{G}$

Lemma $M = N \Rightarrow \underline{\mathrm{Hom}}(N, M)$ has the structure of a coalgebra 1-morphism

Theorem (MMMT)
$\mathbf{M}$ transitive, $\mathscr{C}$ fiat, $N \neq 0$. Then

1) $\underline{\mathbf{M}} \simeq \mathrm{comod}_{\underline{\mathscr{C}}}(A^N)$, $\quad A_N = \underline{\mathrm{Hom}}(N, N)$
2) $\mathbf{M} \simeq \mathrm{injcomod}_{\underline{\mathscr{C}}}(A^N)$

Consequence: classifying simple transitive 2-reps of $\mathscr{C}$ follows from the classification of "simple" coalgebra objects in $\mathscr{C}$

$\rightsquigarrow$ leads to an alternative proof of the Mackaay–Tubbenhauer theorem:

$$\underbrace{\text{Reps of affine } \mathfrak{sl}_2 \text{ of level } n-2}_{\text{Classified by Ostrik}} \xrightarrow[\sim]{\text{Finkelberg}} \text{2-colour Temperley-Lieb 2-cat.} \xrightarrow{\text{Elias}} \text{Singular Soergel bimodules}$$

$$\text{Singular Soergel bimodules} \xrightarrow{\Theta^{out}} \text{Soergel bimodules}$$

$$\Downarrow$$

$$A \rightsquigarrow \hat{A} \in \text{Soergel bimodules}$$

Another application: 2-categories

A, B $\Bbbk$-algebras $B \subseteq A$ $\mathscr{A}$ a subcategory of $\mathscr{C}$

$\Rightarrow$ $A\text{-mod} \underset{\mathrm{Ind}_B^A \simeq A\otimes_B -}{\overset{\mathrm{Res}_B^A}{\rightleftarrows}} B\text{-mod}$ $\qquad$ $\mathscr{C}\text{-fmod} \underset{???}{\rightleftarrows} \mathscr{A}\text{-fmod}$

Cheating: assume $\mathscr{C}$ is fiat and that $\mathscr{A}$ is a fiat subcategory

$\mathbf{M}$ transitive $\simeq$ $\mathrm{comod}_{\mathscr{A}}(A^N)$

algebra object in $\underline{\mathscr{A}}$

$\cap$

$\mathrm{comod}_{\mathscr{C}}(A^N)$ $\leftarrow$ 2-rep of $\mathscr{C}$

Corollary If $\mathscr{C}$ is fiat and $i \in \mathscr{C}$ then there is a bijection, up to equivalance,

$$\mathscr{C}(i,i) \longrightarrow \left\{\begin{array}{l}\text{simple trans. 2-reps of } \mathscr{C} \text{ that} \\ \text{don't vanish on } i\end{array}\right\}$$