Outlook: Category $\mathcal{O}$

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Cat. $\mathcal{O}$ is defined by Bernstein-Gelfand[2] (BGG) in the early 1970's, originally motivated by Verma's work on the determination of the composition factors and their multiplicities.

Recall from the past talks: $\mathcal{U}(\mathfrak{g})$-Mod is a too huge category to be described algebraically.

Last time we introduced the philosophy:

"Break things into smaller pieces."

$$\mathcal{W} \simeq \bigoplus_{\xi \in \mathbb{C}/2\mathbb{Z}} \mathcal{W}^{\xi} \simeq \bigoplus_{\substack{\xi \in \mathbb{C}/2\mathbb{Z} \\ \tau \in \mathbb{C}}} \underbrace{\mathcal{W}^{\xi,\tau}}_{\text{blocks}}$$

cat of weight modules

Good, but lacks some properties!

We found the simples:

Given $p(\mu) = \tau - (\mu+1)^2 = 0$

Cases:

- $p(\mu)$ has no roots: $\mathcal{W}^{\xi,\tau}$ has only one simple $M(\xi,\tau)$
- $p(\mu)$ one root: two simples, $M(\lambda)$ and $\overline{M}(\lambda+2)$
- $p(\mu)$ two roots: three simples, namely $V^{(n)}$, $M(-n-2)$ and $\overline{M}(n+2)$

We keep following the same strategy and try to extract more information about the $\mathfrak{g}$-modules by introducing some "finiteness" conditions, coming from the structure of category $\mathcal{O}$.

1. The basics

We introduce a full subcategory of $\mathcal{U}(\mathfrak{g})$-Mod with:

$$\mathrm{Obj}(\mathcal{O}) := \left\{ M \in \mathcal{U}(\mathfrak{g})\text{-Mod} \;\middle|\; \begin{array}{l} \text{(I) } M \text{ is finitely generated} \\ \text{(II) } M \text{ is a weight module, i.e., } M = \bigoplus_{\lambda \in \mathbb{C}} \underbrace{M_\lambda}_{\text{weight space}} \\ \text{(III) } \dim \mathbb{C}[e]v < \infty \text{ for all } v \in M \end{array} \right\}$$

$\mathrm{Mor}(\mathcal{O}) := \{\text{all possible } \mathfrak{g}\text{-homomorphisms}\}$

Rmk: Condition (III) translates as:

$\mathbb{C}[e] \curvearrowright M$ is locally finite, i.e., $\mathbb{C}[e]$ acts via nilpotent matrices $\begin{pmatrix} 0 & * & * \\ & \ddots & * \\ 0 & & 0 \end{pmatrix}$.

We already know some examples.

Exp. 1: The fin. dim. modules $V^{(n)}$, $n\in\mathbb{N}$ that we saw in Talk 1.

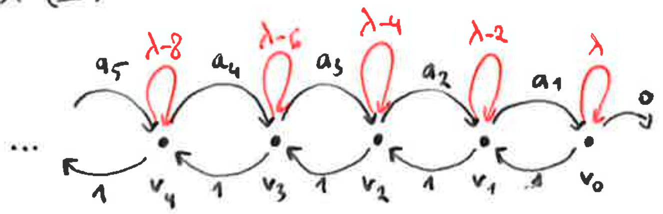
Here (I) and (III) are clear.

(II) follows from the classification theorem (Talk 1) and Weyl's theorem (Talk 2).

Exp. 2: For every $\lambda\in\mathbb{C}$, the Verma module $M(\lambda)\in\mathcal{O}$.

(I) and (II) follow from the definition.

For (III):



$\{v_i : i\in\mathbb{N}_0\}$ basis of $M(\lambda)$

$a_i := i(\lambda - i + 1)$

Let $v = \sum_{i=0}^{k} a_i v_i \in M(\lambda)$.

For every $m\in\mathbb{N}$, the element $e^m(v)$ is a lin. combination of $v_0,\dots,v_k$. Expressed as a matrix:

nilpotent $k\times k$ block

∞-large matrix →

$$\begin{pmatrix} \underbrace{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \end{matrix}}_{k}\Bigg\}k & \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim \mathbb{C}[e](v) \leq k < \infty \Rightarrow M(\lambda)\in\mathcal{O}$.

Some basic properties of cat. $\mathcal{O}$:

Prop: ① $\mathcal{O}$ is closed wrt. taking submodules, quotients and finite direct sums.

② $\mathcal{O}$ is abelian, Krull-Schmidt with usual kernels and cokernels.

③ Simple objects in $\mathcal{O}$ are the simple $\mathfrak{g}$-modules.

④ $\mathcal{O}$ is Artinian and Noetherian.

⑤ $\mathcal{O}$ is not monoidal, i.e., $M(\lambda)\otimes M(\mu)\notin\mathcal{O}$.

⑥ $M\in\mathcal{O}$ and V fin. dim., $M\otimes L\in\mathcal{O}$.

⑦ $\overline{M}(\lambda)\notin\mathcal{O}$.

$\dim\mathbb{C}[e](v) = +\infty$

Prop: The simple objects of $\mathcal{O}$ are simple highest weight modules $L(\lambda)$, $\lambda\in\mathbb{C}$, namely the quotients of the Verma modules.

Rmk: $\mathcal{O}$ is not closed wrt. extensions.

$\overline{\mathcal{W}}$ full subcat. of $\mathcal{W}$, consisting of all weight modules with f.d. weight spaces.

$\xi\in\mathbb{C}/2\mathbb{Z}$ and $\tau\in\mathbb{C}$, set $\mathcal{O}^{\xi,\tau} = \mathcal{O}\cap\overline{\mathcal{W}}^{\xi,\tau}$.

Thm: We obtain the decomposition:

$$\mathcal{O} = \bigoplus_{\substack{\xi\in\mathbb{C}/2\mathbb{Z}\\ \tau\in\mathbb{C}}} \mathcal{O}^{\xi,\tau}$$

abelian categories, called blocks.

Cor: Every object $M \in \mathcal{O}$ has a finite length, i.e. the Jordan-Hölder series are finite:
$0 \subset M_1 \subset M_2 \subset \dots \subset M_n = M$, s.t. M_{i+1}/M_i is simple.

Thm: Let $\xi \in \mathbb{C}/2\mathbb{Z}$, $\tau \in \mathbb{C}$. We have the following result:
1) $\mathcal{O}^{\xi,\tau}$ is trivial $\Leftrightarrow$ $\tau \neq (\lambda+1)^2$ for all $\lambda \in \xi$.
2) There exists a unique $\lambda \in \xi$ s.t. $\tau = (\lambda+1)^2$, then the category $\mathcal{O}^{\xi,\tau}$ has exactly one simple object, namely $M(\lambda) = L(\lambda)$.
3) If there exist $\lambda_1, \lambda_2 \in \xi$ s.t. $\lambda_1 \neq \lambda_2$, $\tau = (\lambda_1+1)^2$ and $\tau = (\lambda_2+1)^2$, then $\lambda_1 \in \mathbb{Z}$ and $\mathcal{O}^{\xi,\tau}$ has two non-isomorphic simple objects $L(\lambda_1)$ and $L(\lambda_2)$.

2. Projective modules

Category $\mathcal{O}$ has an important property that the categories $\widetilde{\mathcal{W}}^{\xi,\tau}$ don't have, namely the existence of a projective cover for each module.

Lemma: For $\lambda \in \mathbb{C} \setminus \{-2,-3,\dots\}$.
① $M(\lambda)$ is projective in $\mathcal{O}$.
② The simple $L(\lambda)$ has a proj. cover in $\mathcal{O}$ (as a quotient of $M(\lambda)$).
③ Let $n \in \{2,3,4,\dots\}$. The simple $L(-n)$ has a proj. cover in $\mathcal{O}$.

Thm: The category $\mathcal{O}$ has enough projective objects.

Proof: Using the fact that every object in $\mathcal{O}$ has finite length, we can prove the existence of a projective cover for $M \in \mathcal{O}$ by induction on the length of M.
Two cases:
- M is simple, then ② and ③ imply the result.
- M is not simple and then consider:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] \arrow[d] & M \arrow[r, two heads] & N \arrow[r] \arrow[d] & 0 \\ & \text{simple} & & \ell(N) < \ell(M) & \end{tikzcd}$$

Let P and Q be projective covers, which exist by assumption:

$$\begin{tikzcd} & P \arrow[d] & P\oplus Q \arrow[l] \arrow[r] \arrow[d, Rightarrow] & Q \arrow[d] \arrow[dl, dashed] & \\ 0 \arrow[r] & L \arrow[r, hook] & M \arrow[r, two heads] & N \arrow[r] & 0 \end{tikzcd}$$

Lift $Q \twoheadrightarrow N$ to a homom. $Q \to M$ s.t. the diagram commutes.
$\Rightarrow$ $P\oplus Q \overset{\subset \text{proj}}{\twoheadrightarrow} M$ is a proj. cover. $\square$

<u>Rmk</u>: The dual statement holds true, namely $\mathcal{O}$ has enough injectives.

Thm. + Rmk. $\Rightarrow$ for every $\lambda \in \mathfrak{h}^*$ we have the indecomposable proj. cover of $L(\lambda)$, denoted as $P(\lambda)$, and the indecomposable injective envelope $I(\lambda)$ of $L(\lambda)$.

3. Blocks and quivers

The previous results imply that every block of $\mathcal{O}$ is an abelian category with enough proj. objects, where all objects have finite length.

<u>Fact</u>: Every block $\mathcal{O}^{\xi,\tau}$ has only finitely many simple objects (up to an iso). This gives us the idea to study blocks via combinatorial tools, called quivers.

<u>Recall</u>: A quiver is a directed graph, consisting of sets of vertices and arrows.

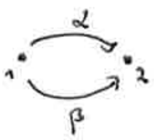
Examples: Path algebra:

① $\bullet$ the only path here is $\{e_1\}$ — trivial, "don't move" — $\mathbb{C}$

②  We have a loop $\Rightarrow$ ∞-dim path algebra $\{e_1, \alpha, \alpha^2, \alpha^3, \ldots\}$ — $\mathbb{C}[\alpha]$

③  $\{e_1, \alpha, \beta, \gamma, \beta\alpha, \ldots\}$ — $\mathbb{C}[\alpha, \beta, \gamma]$

④ $\bullet \quad \bullet \quad \bullet$ $\{e_1, e_2, e_3\}$ — $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$

⑤  $\{e_1, e_2, \alpha, \beta\}$ — Matrices $\begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C}^2 & \mathbb{C} \end{pmatrix}$

To find the simples and the projective covers consider the following ideas:

Projective cover: Find all paths ending at a given vertex.

For example ④ all simples are:

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ L_1 \cong \mathbb{C} & L_2 \cong \mathbb{C} & L_3 \cong \mathbb{C} \\ \cong & \cong & \cong \\ P_1 = \{e_1\} & P_2 = \{e_2\} & P_3 = \{e_3\} \end{array}$$

! Different copies of $\mathbb{C}$

② Simples in ②: only one simple $\{e_1\} \cong \mathbb{C}$.

Proj. cover: $\{e, \alpha, \alpha^2, \ldots\} \cong \mathbb{C}[\alpha]$

For ex.

3. We have two vertices $\Rightarrow$ two simples

Simples: $\mathbb{C}$ corr. to vertex 1

$\mathbb{C}$ corr. to vertex 2

Projectives: $P_1 = \{e_1\} \cong \mathbb{C}$

$P_2 = \{e_2, \alpha, \beta\}$

Advantage of quiver: one can read off all the useful information from the quiver.

Thm: (Description of the blocks of $\mathcal{O}$)

Let $\xi \in \mathbb{C}$ and $\tau \in \mathbb{C}$.

(i) If $(\lambda+1)^2 \neq \tau$ for all $\lambda \in \xi$, then the block $\mathcal{O}^{\xi,\tau}$ is zero. (contains only $\{0\}$)

(ii) If $(\lambda+1)^2 = \tau$ for a unique $\lambda \in \xi$, then the block $\mathcal{O}^{\xi,\tau}$ is semi-simple and $\mathcal{O}^{\xi,\tau} \simeq \mathbb{C}$-mod (or $\mathbb{C}$-vector spaces). equivalence of categories

(iii) If $(\lambda_1+1)^2 = (\lambda_2+1)^2 = \tau$ for $\lambda_1, \lambda_2 \in \xi$, $\lambda_1 \neq \lambda_2$, then $\tau = n^2$ for some $n \in \mathbb{N}$ and the block $\mathcal{O}^{\xi,\tau}$ is equivalent to the category A-Mod (as a $\mathbb{C}$-category) for A a fin. dim. algebra (path algebra).

The corresponding quiver is
$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left, "a"] & \underset{2}{\bullet} \arrow[l, bend left, "b"] \end{tikzcd}$$
with relation $ab = 0$.

Rmk: The quiver in case (ii) is just $\bullet$

$\mathbb{C}$-Mod ($\mathbb{C}$-vector spaces)

$Obj(\mathbb{C}\text{-Mod}) := \{V \text{ vector spaces}\}$

$\mathbb{C} \curvearrowright V$ by scalar

One simple module $\mathbb{C}$ of dim 1. The path algebra is just $\mathbb{C}$.

Rmk: The block $\mathcal{O}_0 := \mathcal{O}^{2\mathbb{Z},1}$ is called principal block and contains two simples, namely $L(0)$ and $L(-2)$. It contains a copy of the trivial representation. For more about it, check the example on the last page of these notes.

Thm: (BGG-reciprocity) Let $\lambda, \mu \in \mathbb{C}$. Denote by $[P(\lambda):M(\mu)]$ the multiplicity of $M(\mu)$ as a subquotient of the filtration of $P(\lambda)$. We have the following result: $[P(\lambda):M(\mu)] = [M(\mu):L(\lambda)]$.

An explicit example:

Consider the principal block $\mathcal{O}_0$, so we are in case (iii) of the theorem about blocks and quivers.

We want to confirm $\mathcal{O}_0 \simeq A$-modules

two simples $L(0)$ and $L(-2)$; $L(0) = L_1$, $L(-2) = L_2$

we need two simples

$$A: \begin{tikzcd} \bullet_1 \arrow[r, bend left, "a"] & \bullet_2 \arrow[l, bend left, "b"] \end{tikzcd}$$

Relation: $ab = 0$

All possible paths are:

e_1

e_2

$a : 1 \to 2$

$b : 2 \to 1$

$ba : 2 \to 1 \to 2$

ex: $ba\underline{b}a = 0$

$a^2 = 0 = b^2$

Simples: $\boxed{\bullet_1}$ and $\boxed{\bullet_2}$, but different copies of $\mathbb{C}$.

$\boxed{\bullet_1} \cong \mathbb{C}$, $\boxed{\bullet_2} \cong \mathbb{C}$

What are the submodules:

Idempotents: $e_1 L_1 = 1$, $e_2 L_1 = 0$, $e_1 L_2 = 0$, $e_2 L_2 = 1$

$L_1 = \{e_1\}$

$L_2 = \{e_2\}$

P_1 has all paths ending at 1

P_2 has all paths ending at 2

$P_1 = \{\overset{L_1}{e_1}, \overset{L_2}{b}\}$ – proj. cover

$P_2 = \{\overset{L_2}{e_2}, \overset{L_1}{a}, \overset{L_2}{ba}\}$ – proj. cover

We notice that $P_1 \not\cong I_1$ (not symmetric!)

$$\begin{tikzcd} L_1 \arrow[d, dash] \\ L_2 \end{tikzcd}$$

for $\lambda = 0$

$P_2 \cong I_2$ (symmetric picture!)

$$\begin{tikzcd} L_2 \arrow[d, dash] \\ L_1 \arrow[d, dash] \\ L_2 \end{tikzcd}$$

for $\lambda = -2$

$$\text{Cartan matrix } C(\mathcal{O}_0) := \begin{array}{c c} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{c} L_1 \\ L_2 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \end{array}$$

times L_1 occurs in the filtration P_1, similarly for the others

By BGG: $[P:L] = [P:M][M:L]$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ (here $C = DD^T$)

$\det(C)$ measures how far from semi-simplicity our module category is, in our case it is $\det(C) = 1$.

$\Rightarrow$ $\mathcal{O}_0$ is close to semi-simple