Talk 3: Universal enveloping algebra 29.04.2019

The Harish-Chandra Homomorphism (and some algebra)

Goals for today: ① Continue studying the center of $Z(\mathcal{U}(g))$;
② Recall/Define some algebraic facts about $\mathcal{U}(g)$.

From Daniel's talk last week:

We work with $\mathcal{U}(g) = \mathbb{C}\langle e,f,h\rangle/\text{Relations}$, we have the PBW basis $\{f^i h^j e^k \mid i,j,k \in \mathbb{N}_0\}$.
We defined: • filtration $\mathcal{U}(g)^i$;
• grading $\mathcal{U}(g)_i$.

$\left.\begin{array}{l}\deg(e)=1\\ \deg(h)=0\\ \deg(f)=-1\end{array}\right\}$ choice of degree

We identified the centralizer $\mathcal{U}(g)_0 \cong \text{span}\langle h,c\rangle \cong \mathbb{C}[h,c]$, which is a free $Z(g)$-module with basis $\{1,h,h^2,\dots\}$

$h \in \mathcal{U}(g)_0$ "polynomials of deg 0"

Identified the center of $\mathcal{U}(g)$, namely $Z(\mathcal{U}(g)) = \mathbb{C}[c]$ ↪ generated by $c=(h+1)^2+4fe$
$\cap$
$\underbrace{\mathcal{U}(g)_0}$
centralizer of $\mathfrak{h}$ → Cartan subalgebra

In Samuel's talk we saw that $C_{V^{(n)}} = n^2 \cdot \text{id}_{V^{(n)}}$ for all $n \in \mathbb{N}$ and $V^{(n)}$ is a simple finite-dimensional g-module.

Q: How to find the scalar n?

Take a highest vector v_0 for which we know that $E(v_0)=0$ and $H(v_0)=(n-1)v_0$.

We will see that the action of an element of $Z(g)$ reduces to an action of an element from $\mathbb{C}[h]$, namely $c(v_0) = (h+1)^2(v_0)$
↳ $((h+1)^2 + \underbrace{4fe}_{=0})$

Lemma: (i) The sets defined as $I = \overbrace{\mathcal{U}(g)e}^{\text{left ideal generated by } e} \cap \mathcal{U}(g)_0$ and $I = \overbrace{f\,\mathcal{U}(g)}^{\text{right ideal generated by } f} \cap \mathcal{U}(g)_0$ coincide, i.e. I is an ideal of $\mathcal{U}(g)_0$.

(ii) $\mathcal{U}(g)_0 = \mathbb{C}[h] \oplus I$. "I is a two-sided ideal, complementary to $\mathcal{U}(\mathfrak{h})$"

Proof: (i) Every element in I is a linear combination of some elements of the form ue, where u is some standard monomial, i.e. u is of the form $f^i h^j e^k$.

Take $\underset{\deg -1}{u}\,\underset{\deg 1}{e} \in \mathcal{U}(g)_0 \iff u = f^{i+1} h^j e^i$ for some $i,j \in \mathbb{N}_0$.
"need $\deg(u) = -1$"

Then $ue = f(f^i h^j e^{i+1}) \in f\mathcal{U}(g) \Rightarrow I \subset f\mathcal{U}(g) \cap \mathcal{U}(g)_0$.

The opposite inclusion $I \supset f\mathcal{U}(g) \cap \mathcal{U}(g)_0$ follows similarly.

(ii) Notice that $I \cap \mathbb{C}[h] = 0$.

By the PBW Thm. for $\mathcal{U}(g)_0$ we get that $\mathcal{U}(g)_0 = \mathbb{C}[h] \oplus I$. $\square$

def: Let $\kappa : \mathcal{U}(g)_0 \twoheadrightarrow \underset{\sim}{\mathbb{C}[h]}^{\mathcal{U}(h)}$ be a projection with $Ker(\kappa) = I$ from the definition. κ is a homomorphism of associative algebras and is called the Harish-Chandra homomorphism.

Main property:

Prop: Let V be a g-module and $v \in V$ an element of highest weight s.t. $E(v) = 0$. Then for any $g \in Z(g)$ we have $g(v) = \kappa(g) \cdot v$.

"With other words the projection κ restricts to $\kappa' : Z(g) \to \mathbb{C}[h]$."

Proof: From the last time we know that $Z(g) = \mathbb{C}[c]$ and hence $g = g(c) \in \mathbb{C}[c]$. $c = (h+1)^2 + 4fe$ and the def. of v, i.e. $E(v) = 0$ and using the definition of κ:

$$c(v) = (H+1)^2(v) = \kappa(c)(v)$$

Since κ is a homomorphism $\Rightarrow g(c)(v) = \kappa(g(c)) \cdot v$ as well, so we have the restriction. $\square$

The restriction can be very useful for the study of the g-modules:

Thm: $\hat{\kappa} : Z(g) \xrightarrow{\sim} \mathbb{C}[(h+1)^2]$ turns out to be an isomorphism.

Proof: "Nothing to prove".
follows from the definition

Rmk: The object $\mathbb{C}[(h+1)^2]$ is interesting, since it has a nice interpretation in terms of invariant polynomials.

A polynomial f is invariant, if $f(x) = f(\gamma x)$, for $\gamma \in G$, i.e. the polynomial doesn't change, if a group acts on it.

"3-dim rep. of $\mathfrak{g}$ acting on itself"

Recall the adjoint action $ad: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.

The adjoint action of h, $[h,-]$ is diagonalizable, namely we have:

$[h,f] = -2f$

$[h,h] = 0$

$[h,e] = 2e$

The set of eigenvalues $\{\underset{-\alpha}{-2}, \underset{\alpha_0}{0}, \underset{\alpha}{2}\}$.

Consider $\{-2,0,2\}$ as elements of the dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$.

We call the non-zero eigenvalues "roots" and denote them as $\Delta = \{-2,2\}$.

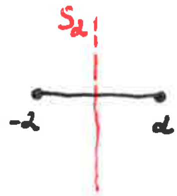
Distinguish the positive roots: $\Delta^+ = \{2\}$.

The set of all non-zero eigenvalues is called a root system of $\mathfrak{g}$.

Let $W \subset GL(\underset{\mathbb{C}}{\underset{\shortparallel}{\mathfrak{h}^*}})$ preserving the set of the eigenvalues.

Weyl group of $\mathfrak{g}$

We can depict the situation as follows:



Type $A_1 = sl_2(\mathbb{C})$

↳ generated by reflections s_α

In our case $W \cong S_2 = \{$identity, multiplication by $-1\}$
= swapping the roots

$W \curvearrowright \mathfrak{h}^* \xrightarrow[\text{induces}]{\text{naturally}} W \curvearrowright \mathfrak{h} \implies \mathbb{C}[h] =$ algebra of polynomial functions on $\mathfrak{h}^*$
$\mathbb{C}[\mathfrak{h}^*]$

Let ρ be the half of the sum of the positive roots, in our case it is $\rho = 1 = \frac{1}{2} \cdot 2 \in \mathbb{C} = \mathfrak{h}^*$, since $\Delta^+ = \{2\}$.

Denote by $\gamma: \mathbb{C}[h] \to \mathbb{C}[h]$ automorphism
$\quad g \mapsto (\lambda \mapsto g(\lambda - \rho))$

a polynomial function on $\mathfrak{h}^*$

"translation by -1 in our case?"

A direct application of this theory:

<u>Cor:</u> The restriction $\gamma\circ\kappa\big|_{Z(g)} : \mathcal{U}(g)_0 \xrightarrow{\kappa} \mathbb{C}[h] \xrightarrow{\gamma} \mathbb{C}[h]^W$ is an isomorphism.
$g \longmapsto \gamma(g)$

This shortens to $Z(g) \xrightarrow{\sim} \mathbb{C}[h]^W$, where $\mathbb{C}[h]^W$ is the algebra of polynomials, invariant with respect to the action of W.

<u>Proof:</u> From the previous thm. we have $\kappa\big|_{Z(g)} : Z(g) \xrightarrow{\sim} \mathbb{C}[(h+1)^2]$.

For any polynomial $g \in \mathbb{C}[(h+1)^2]$ we have:

In our case the action of W is just swapping h and $-h$.

$$\gamma(g)((h+1)^2) = g((\underset{\overset{\shortparallel}{\rho}}{h+1-1})^2) = g(h^2)$$

Since $\mathbb{C}[h^2] = \mathbb{C}[h]^W$ algebra of invariant polynomials $\Rightarrow$ the claim follows. $\square$

General result:

$$\boxed{\begin{array}{l} Z(\mathcal{U}(g)) \cong \mathbb{C}(\mathfrak{h})^W \\ \quad \#\text{variables} = \dim(\mathfrak{h}) \\ \qquad\qquad\qquad \hookrightarrow \text{Cartan subalg.} \end{array}}$$

Some algebra:

def: A ring R is called Noetherian, if every ascending chain of ideals

$I_1 \subset I_2 \subset I_3 \subset \ldots \subset R$ stabilizes, i.e. $\exists n \in \mathbb{N}$ s.t. $I_n = I_{n+k}$ for all $k \in \mathbb{N}$.

Equivalently: a ring R is called Noetherian, if every ideal I is finitely generated.

Rem: Finitely generated means that $\exists a_1, \ldots, a_n \in R$ generators s.t. $I = (a_1, \ldots, a_n)$
$\qquad = \{x_1 a_1 + \ldots + x_n a_n : x_i \in R\}$

def: A ring R is called Artinian, if every descending chain of ideals

$I_1 \supset I_2 \supset I_3 \supset \ldots$ stabilizes, i.e. $\exists n \in \mathbb{N}$ s.t. $I_n = I_{n+k}$ for all $k \in \mathbb{N}$.

Remark: Artinian $\Rightarrow$ Noetherian
$\not\Leftarrow$

Examples: • Noetherian: $\mathbb{Z}, \mathbb{C}, \mathbb{C}[X], \mathbb{Z}[i]$, $\mathbb{C}[X_1, \ldots, X_n]$. → fin. many!

Non-example: $\mathbb{C}[X_1, X_2, \ldots]$: → ∞-many

$\langle X_1 \rangle \subset \langle X_1, X_2 \rangle \subset \langle X_1, X_2, X_3 \rangle \subset \ldots$ never terminates!

Non-Artinian: $\mathbb{Z}$, $\mathbb{C}[X_1, \ldots, X_n]$

$\langle 2 \rangle \supset \langle 4 \rangle \supset \langle 8 \rangle \supset \ldots$ never stabilizes

$X \supset X^2 \supset X^3 \supset \ldots$ same logic.

Side remark: $I = (x, x^2, x^3, \ldots) = (x)$
$I = (2, 3, 5, 7, \ldots) = (1)$ ← $\boxed{\text{apply Euclidean algorithm } -2+3=1}$

Noetherian ring is a ring where we can perform Euclidean algorithm.

$\boxed{\text{Hilbert's Basis Thm: If } R \text{ is a Noetherian ring} \Rightarrow R[X] \text{ is also Noetherian.}}$

Thm: $\mathcal{U}(\mathfrak{g})$ is a left Noetherian and a right Noetherian.

Rem: But it is <u>not</u> Artinian.

Proof: Take the associated graded algebra $G(\mathcal{U}(\mathfrak{g})) \cong \mathbb{C}[e, f, h]$.
Apply the Hilbert's basis Thm. □

$\mathcal{U}(\mathfrak{g})$ is "a polynomial ring"