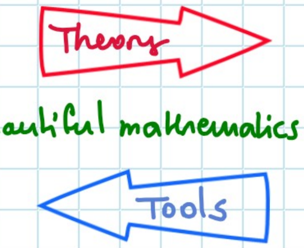
Representation theory of the Symmetric Group S_n

The aim of today's talk is to give a very brief introduction to the representation Cayley's theorem theory of a particular finite group, namely the symmetric group S_n.
It turns out that the representation theory of the symmetric group has a nice combinatorial counterpart.

Representation theory

- representations
- simples
- characters
- dimensions
- multiplicities

⋮



Combinatorics

- tableaux
- algorithms
- algorithms
- algorithms

⋮ (computations)

1. Combinatorial preliminaries

def: A partition of $n \in \mathbb{Z}_{\geq 0}$ is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell$ and $\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_\ell = n$. Notation: $\lambda \vdash n$.

Exp: $n=4$, all possible partitions are
(4)
$(3,1)$
$(2,2)$
$(2,1,1)$
$(1,1,1,1)$

$(1,2,1)$ is not a partition!

To any $\sigma \in S_n$ we associate a natural partition called the cycle type of σ.
We define it as $t(\sigma) = (\lambda_1, \dots, \lambda_\ell)$, where λ_i are the lengths of the cycles of σ in decreasing order (in one-line notation) including the multiplicities and the cycles of length 1.

Exp: $\sigma \in S_8$

$\sigma = (2486)(35)(1)(7)$ $\qquad$ $t(\sigma) = (4,2,1,1)$

Recall the following fact from linear algebra.

Thm: Let $\sigma, \tau \in S_n$. Then σ is conjugate to τ, if and only if $type(\sigma) = type(\tau)$.

Thus, we can make the following observation:

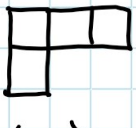
$$\left\{ \begin{array}{c} \text{Conjugacy} \\ \text{classes of } S_n \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Partitions} \\ \text{of } n \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{simples} \\ \text{of } S_n \end{array} \right\}$$

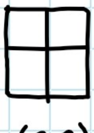
Our goal for today

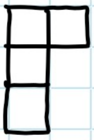
def: A <u>Young diagram</u> is a finite collection of boxes arranged in left-justified rows, with the row sizes weakly decreasing. The Young diagram associated to a partition $\lambda = (\lambda_1, \lambda_2, .., \lambda_l)$ is the one that has l rows, and λ_i boxes in the i^{th} row.

Exp:


(4)


(3,1)

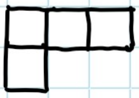

(2,2)


(2,1,1)

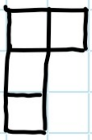

(1,1,1,1)

The <u>conjugate partition</u> λ^T of λ is the partition whose Young diagram is the transpose of the diagram of λ, i.e. λ^T is obtained by exchanging rows and columns in λ.

Exp: If $\lambda =$


(3,1)

, then $\lambda^T = (2,1,1)$



We want to define a partial order on partitions.

def: Suppose that $\lambda = (\lambda_1, .., \lambda_l)$ and $\mu = (\mu_1, .., \mu_m)$ are partitions of n. Then λ is said to <u>dominate</u> μ, if

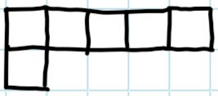
$$\sum_{i=1} \lambda_i \geq \sum_{i=1} \mu_i$$

(Notation: $\lambda \trianglerighteq \mu$)

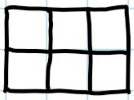
for all $i \geq 1$ where if $i > l$, then we take $\lambda_i = 0$, and if $i > m$, then we take $\mu_i = 0$.

Rmk: The dominance order defines an equivalence relation.

Exp: Let $\lambda = (5,1)$ and $\mu = (3,3)$. Then $(5,1) \trianglerighteq (3,3)$.



$\trianglerighteq$

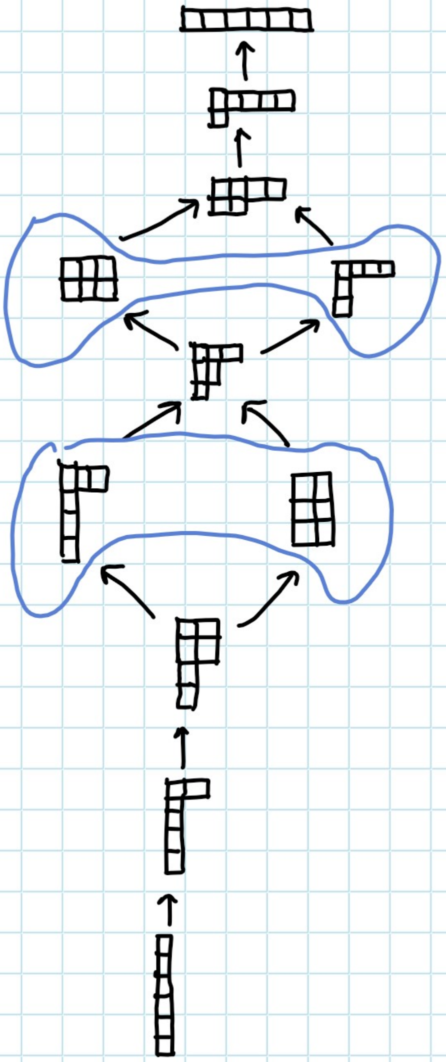


$5 > 3$

$5+1 \geq 3+3$

Note that some partitions can't be compaired, for instance $(3,1,1,1)$ and $(2,2,2)$. Dominance relations are depicted via Hasse diagrams.

Let $n=6$:



can't be compaired

$3 < 4$

$3+3 > 4+1$

can't be compaired

$3 > 2$

$3+1 < 2+1$

$$\left(\begin{array}{c} A \\ \uparrow \\ B \end{array} \quad \text{reads as } A \trianglerighteq B \right)$$

def: Let $\lambda \vdash n$. A Young tableau t of shape λ is obtained by filling in the boxes of a Young diagram of λ with $1,2,\dots,n$, with each number occuring exactly once. In this case we say that t is a λ-tableau. There are clearly $n!$ λ-tableaux.

A standard Young tableau (SYT) is a Young tableau whose entries are strictly increasing across each row and each column.

Exp: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ $\quad\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$

↑ SYT   ↑ SYT

A semistandard Young tableau (SSYT) has entries that are weakly increasing across each row and strictly increasing down each column, i.e. may have repeated entries in the rows.

Exp:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \nwarrow \text{SSYT}$$

"SYT ⇒ SSYT but not the opposite"

There exists a useful criterion for domination.

Dominance lemma: Let λ and μ be partitions of n and suppose that t^λ and s^μ are tableaux of shape λ and μ. The integers in the same row of s^μ are located in different columns of t^λ. Then $\lambda \trianglerighteq \mu$.

Exp: $\lambda = (4,3,1)$ and $\mu = (3,2,2,1)$.

$$t^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 5 & 3 & 6 \\ \hline 4 & 2 & 7 \\ \cline{1-3} 8 \\ \cline{1-1}\end{array} \qquad s^\mu = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \qquad \lambda \trianglerighteq \mu$$

2. The simples of S_n

Let $X \subseteq \{1,\dots,n\}$, we identify S_X with those permutations in S_n that fix all the elements outside of X, e.g. $S_{\{2,3\}} = \{id, (23)\}$.

Thus, we shall have a look at two interesting subgroups of S_n.

def: For a tableau t of size n, the <u>row group</u> of t, denoted R_t, is the subgroup of S_n consisting of permutations which only permute the elements within each row of t. Similarly, the column group C_t is the subgroup of S_n consisting of permutations which only permute the elements within each column of t.

Exp:

$$t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \qquad R_t = S_{\{1,2,4\}} \times S_{\{3,5\}} \qquad |R_t| = 3!\,2! = 12$$

$$C_t = S_{\{3,4\}} \times S_{\{1,5\}} \times S_{\{2\}} \qquad |C_t| = 2!\,2! = 4$$

We define an action of S_n on the set of λ-tableaux by applying $\sigma \in S_n$ to the entries of the boxes.

Exp:

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \sigma = (123), \quad \sigma t = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$$

We define an equivalence relation $\sim$ on the set of λ-tableaux by saying that $t_1 \sim t_2 \Leftrightarrow \exists\, \sigma \in R_{t_1}$ s.t. $\sigma t_1 = t_2$, i.e. the rows of t_1 and t_2 have the same elements.

Exercise: Check that $\sim$ is an equivalence relation.

__def:__ An $\sim$equivalence class of λ-tableaux is called a __λ-tabloid__ or a __tabloid of shape λ__. The tabloid of a tableau t is denoted as $[t]$. The set of all tabloids of shape λ is denoted T^λ.

__Exp:__ Let $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, then the tabloid $[t]$ is drawn as $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, which represents the equivalence class containing the following two tableaux: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$.

__Q:__ How does S_n act on tabloids?

__Prop:__ Suppose that $t_1 \sim t_2$ and $\sigma \in S_n$. Then $\sigma t_1 = \sigma t_2$. Hence there is a well-defined action of S_n on T^λ given by $\sigma[t] = [\sigma t]$ for t a λ-tableau.

__Exp:__ $\sigma = (123) \in S_3$

$$(123)\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$$

"σ gives the instruction how to move from one row to another"

We saw how S_n acts on the set of tabloids, so we are ready to define a representation.

__def:__ For a partition λ, set $M^\lambda = \mathbb{C}T^\lambda$, a vector space spanned by T^λ.
We call $\varphi^\lambda : S_n \to GL(M^\lambda)$ the __permutation representation__ associated to λ.

__Exp:__ Consider $\lambda = (n)$. We see that M^λ is the vector space generated by the single tabloid $\begin{array}{|cccc|}\hline 1 & 2 & \cdots & n \\ \hline\end{array}$. This tabloid is fixed by S_n, thus $M^{(n)}$ corresponds to the one-dimensional trivial representation.

__Exp:__ $\lambda = (1^n) = (1,1,1,\dots,1)$. Each $[t]$ consists of one tableau and this tableau can be identified with a permutation. Thus, $M^{(1^n)} \cong \mathbb{C}[S_n]$, which is the regular representation.

Exp: $\lambda = (n-1,1)$

Let $[t_i]$ be the λ-tabloid with i in the second row. Then M^λ has a basis $[t_1], [t_2], \dots, [t_n]$. The action of $\sigma \in S_n$ sends t_i to $t_{\sigma(i)}$. So, $M^{(n-1,1)} \cong$ defining representation of $\mathbb{C}\{1,2,..,n\}$.

Consider $M^{(3,1)}$:

The basis is given by $t_1 = \begin{array}{l}\boxed{2\ \ 3\ \ 4}\\ \boxed{1}\end{array}$, $t_2 = \begin{array}{l}\boxed{1\ \ 3\ \ 4}\\ \boxed{2}\end{array}$, $t_3 = \begin{array}{l}\boxed{1\ \ 2\ \ 4}\\ \boxed{3}\end{array}$, $t_4 = \begin{array}{l}\boxed{1\ \ 2\ \ 3}\\ \boxed{4}\end{array}$.

We can compute the dimension and the characters of the rep. M^λ.

Prop: If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_e)$, then $\dim M^\lambda = \dfrac{n!}{\lambda_1! \lambda_2! \dots \lambda_e!}$.

Prop: Suppose $\lambda = (\lambda_1, \dots, \lambda_e)$, $\mu = (\mu_1, \dots, \mu_m)$ are partitions of n. The character of M^λ evaluated at an element of S_n with cycle type μ is equal to the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_e^{\lambda_e}$ in $\prod_{i=1}^{m} (x_1^{\mu_i} + x_2^{\mu_i} + \dots + x_e^{\mu_i})$.

Exp: Consider S_4. We compute the character table for the permutation rep. for S_4.

permutation cycle type	e $(1,1,1,1)$	(12) $(2,1,1)$	$(12)(34)$ $(2,2)$	(123) $(3,1)$	(1234) (4)
$M^{(4)}$	1	1	1	1	1
$M^{(3,1)}$	4	2	0	1	0
$M^{(2,2)}$	6	2	2	0	0
$M^{(2,1,1)}$	12	2	0	0	0
$M^{(1,1,1,1)}$	24	0	0	0	0

To compute the character of $M^{(3,1)}$ at (12), consider $\lambda = (3,1)$ and $\mu = (2,1,1)$. Compute $\prod_{i=1}^{3} (x_1^{\mu_i} + x_2^{\mu_i}) = (x_1^2 + x_2^2)(x_1 + x_2)^2 =$

$= (x_1^2 + x_2^2)(x_1^2 + 2x_1x_2 + x_2^2) = x_1^4 + 2x_1^3x_2 + x_1^2x_2^2 + x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 =$

$= x_1^4 + \boxed{2x_1^3x_2} + 2x_1x_2^3 + 2x_1^2x_2^2 + x_2^4$.

$\chi_{M^{(3,1)}}(12) =$ coeff. of $x_1^3x_2 = 2$.

Similarly, one computes all the other characters.

⚠ Note that this is not the character table for S_n, since all the M^λ are reducible with exception of $M^{(n)}$!

We are interested in the simples of S_n contained in M^λ. We need to find a strategy how to extract the simples from M^λ.

First, we need to select elements from M^λ to span a subspace of M^λ.

def: If t is a tableau, then the associated polytabloid is $e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\, \pi[t]$.

Exp: $t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}$, then we compute $e_t = \overset{e}{\begin{array}{|ccc|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}} - \overset{(34)}{\begin{array}{|ccc|}\hline 3 & 1 & 2 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}} - \overset{(15)}{\begin{array}{|ccc|}\hline 4 & 5 & 2 \\ \hline 3 & 1 \\ \cline{1-2} \end{array}} + \overset{(34)(15)}{\begin{array}{|ccc|}\hline 3 & 5 & 2 \\ \hline 4 & 1 \\ \cline{1-2} \end{array}}$.

Q: How does S_n act on the set of polytabloids?

Lemma: Let t be a tableau and π be a permutation. Then $e_{\pi t} = \pi e_t$.

Now we are ready to describe the simples contained in M^λ.

def: For any partition λ, the corresponding Specht module, denoted S^λ, is the submodule of M^λ spanned by the polytabloids e_t, where t is taken over all tableaux of shape λ.

Exp: Let $\lambda = (n)$, there is only one polytabloid, namely $\begin{array}{|c|}\hline 1\,2\,3 \cdots\cdots n \\ \hline \end{array}$. It is fixed by S_n, we see that $S^{(n)}$ is the one-dimensional trivial representation.

Exp: Let $\lambda = (1^n) = (1,1,..,1)$.

$t = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$

Note that e_t is the sum of all the λ-tabloids multiplied by the sign of the applied permutation.

For any other $\overset{\lambda\text{-tab.}}{t'}$ we have $e_t = e_{t'}$ (even permutation) or $e_t = -e_{t'}$ (odd perm).

using the Lemma

$\Rightarrow S^{(1^n)}$ is the sign permutation.

Exp: Let $\lambda = (n-1,1)$, again let $[t_i]$ denote the λ-tabloid with i in the second row.

Since we have only two rows, all the polytabloids are of the form $[t_i] - [t_j]$.

$\begin{array}{|c|c|}\hline i & \text{rest...} \\ \hline j \\ \cline{1-1} \end{array}$ Let $e_i = [t_i]$. Then the space S^λ is spanned by the elements of the form $e_i - e_j$, thus $S^{(n-1,1)} = \{c_1 e_1 + c_2 e_2 + \dots + c_n e_n \mid c_1 + c_2 + \dots + c_n = 0\}$.

We obtain the standard representation. The direct sum $S^{(n-1,1)} \oplus S^{(n)} \cong M^{(n-1,1)}$

permutation rep.
(defining rep.)

Rmk: Recall that S_3 has 3 simple reps: the trivial, the sign and the standard. They correspond to partitions (3), (1,1,1) and (2,1). These are exactly the Specht representations. This is a general result!

Thm: The Specht modules S^λ for $\lambda \vdash n$ form a complete list of irreducible reps. of S_n over $\mathbb{C}$.

Q: What is the basis for S^λ?

Thm: Let λ be any partition. The set $\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$ forms a basis for S^λ as a vector space.

Now, a couple of results regarding dimensions of S^λ and its characters.

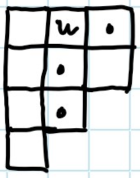
Let $f^\lambda :=$ # number of standard λ-tableaux.

Fact: Suppose $\lambda \vdash n$, then $\dim S^\lambda = f^\lambda$.

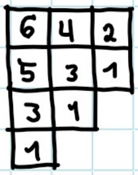
Equivalently, one can use the so-called hook-length formula, found by Frame, Robinson and Thrall.

def: Let λ be a Young diagram. For a box u in the diagram (denoted by $u \in \lambda$), we define the hook of u (or at u) to be the set of all boxes directly to the right of u or directly below u (including u). The number of boxes in the hook is called the hook-length of u, denoted by $h_\lambda(u)$.

Exp: Consider $\lambda = (3,3,2,1)$, $n = 9$



hook at u



hook-lengths

Thm: $\lambda \vdash n$ Young diagram. $\dim S^\lambda = f^\lambda = \dfrac{n!}{\prod_{u \in \lambda} h_\lambda(u)}$.

Exp: $\begin{array}{|c|c|}\hline 3&2\\\hline 2&1\\\hline\end{array}$ $n=4$ $\lambda=(2,2)$ $\dim S^{(2,2)} = \frac{4!}{3.2.2} = \frac{24}{12} = 2$

Q: How to compute the characters of S^λ?

Thm. (Frobenius formula) Suppose $\lambda=(\lambda_1,..,\lambda_\ell)$ and $\mu=(\mu_1,..,\mu_m)$ are partitions of n. The character of S^λ evaluated at an element of S_n with cycle type μ is equal to the coeff. of $x_1^{\lambda_1+\ell-1}x_2^{\lambda_2+\ell-2}\dots x_\ell^{\lambda_\ell}$ in

$$\prod_{1\le i<j\le\ell}(x_i-x_j)\prod_{i=1}^{m}(x_1^{\mu_i}+x_2^{\mu_i}+\dots+x_\ell^{\mu_i}).$$

Exp: Character table of S_3:

permutation cycle type	e (3)	(12) $(2,1)$	(123) $(1,1,1)$
	$\textcircled{1}^{*}$	1	1
	2	$\textcircled{0}^{**}$	-1
	1	-1	1

* Here $\lambda=(3)$
$\mu=(3)$

Look for the coeff. of x_1^3 in

$1.x^3$ $\Rightarrow$ obvious

** Here $\lambda=(2,1)$, $\mu=(2,1)$.
Look for the coeff. of $x_1^3x_2$

in $(x_1-x_2)(x_1^2+x_2^2)(x_1+x_2) =$

$= (x_1^2-x_2^2)(x_1^2+x_2^2) = x_1^4-x_2^4$

$\Rightarrow$ the coeff. is 0

Similarly, one computes the other characters.

Q: How about multiplicities?

Prop: M^μ contains S^λ as a subrep, iff $\lambda\trianglerighteq\mu$. Also, M^μ contains exactly one copy of S^μ.

If we want to answer the question how many copies of S^λ are contained in M^μ one has to consider the SSYT.

def: The content of a SSYT T of shape λ is the composition $\mu=(\mu_1,..,\mu_m)$, where μ_i equals the number of i's in T.

Exp: $\lambda=(4,2,1)$, content $\mu=(2,2,1,0,1,1)$:

$$\begin{array}{cccc} 1 & 1 & 2 & 5 \\ 2 & 3 & & \\ 6 & & & \end{array}$$

def: Suppose $\lambda,\mu \vdash n$, the Kostka number $K_{\lambda\mu}$ is the number of SSYT of shape λ and content μ.

Exp: $\lambda=(3,2)$
$\mu=(2,2,1)$

$$\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \end{array} \quad \text{and} \quad \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & \end{array} \qquad K_{\lambda\mu}=2$$

Prop: Suppose that $\lambda,\mu\vdash n$. Then $K_{\lambda\mu}\neq 0$, iff $\lambda \trianglerighteq \mu$. Also, $K_{\lambda\lambda}=1$.

Thm: (Young's rule) $M^{\mu}\cong \bigoplus_{\lambda\trianglerighteq\mu} K_{\lambda\mu} S^{\lambda}$.

Exp: $M^{(2,2,1)} \cong S^{(2,2,1)} \oplus S^{(3,1,1)} \oplus 2\,S^{(3,2)} \oplus \underbrace{2}\,S^{(4,1)} \oplus S^{(5)}$

$$\begin{array}{cccc} 1 & 1 & 2 & 2 \\ 3 & & & \end{array} \quad \text{and} \quad \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & & & \end{array}$$

Exp: $K_{(n)\mu}=1$ since there is only one (n)-SSYT of content μ.
The Young's rule implies that every M^{μ} contains exactly one copy of the trivial rep. $S^{(n)}$.

Summary of the talk

Representation theory of S_n	Combinatorics of S_n
• conjugacy classes	• Young diagrams (partitions)
• simples	• Specht modules
• characters	• Frobenius character formula
• dimensions	• Hook-length formula
• multiplicities	• Kostka number (SSYT)