Talk 1: Group representations I

23.09.2019

Idea of rep. theory of finite groups:

G := finite group

X := finite set

V := fin. dim. vector space (for us the field will be $\mathbb{C}$)

"group acting on a set" $G \curvearrowright X$ $\xrightarrow{\text{linearization of the group action}}$ "group acting on a vector space" $G \curvearrowright V$

1) Classify fin. sets with G-action;

$\Updownarrow$

2) Classify subgroups $H \subset G$ up to conjugacy.

difficult!

1) Classify $\mathbb{C}$-vector spaces with linear G-action.

easy!

Linear algebra provides us tools.

Remark: Recall that there are two equivalent ways to understand the group action.

① $G \curvearrowright X$ as a map $\begin{array}{l} G \times X \to X \\ (g,x) \mapsto gx \end{array}$ $\left(\begin{array}{l} ex = x \text{ for } e \in G \\ \text{and } (gh)x = g(hx) \text{ for all } g,h \in G \end{array} \right)$

② $G \curvearrowright X$ as a homomorphism $\rho: G \to S_X$, where $S_X := \{ f: X \to X \mid f \text{ is bijective} \}$.

1. The basics

def: A <u>representation</u> of a group G is a (group) homomorphism $\varphi: G \to GL_n(V)$ for V a (fin. dim) vector space.

We call the dimension of V the <u>degree</u> of φ.

Remark: Similarly to the remark above, we can define $\begin{array}{l} G \times V \to V \\ (g,v) \mapsto g.v \end{array}$ which gives us the notion of a G-module V.

The notions of a representation and a G-module are equivalent.

Notation: We denote the representation by φ_g, $\varphi(g)$ or $\varphi_g(v)$, or even $\varphi_g v$ for the action of φ_g on $v \in V$.

Rem: The 0-dim. representations are discarded from consideration.

Examples of 1-dim. reps:

1) For every group G, there exists a representation, given by:
$\varphi: G \to \mathbb{C}^\times$, $\varphi(g) = 1$ for all $g \in G$.
as 1×1-matrix

2) $G = \mathbb{Z}/2\mathbb{Z} = \{[0],[1]\}$, $\varphi: \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$
$[m] \mapsto (-1)^m$ for $[m] \in G$

3) Generally, for a cyclic group $G = \mathbb{Z}/n\mathbb{Z}$, we have $\varphi: \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$
$[m] \mapsto e^{2\pi i m/n}$
for $n = 0,1,\dots,m$. (n^{th} root of unity)

Recall the notion of change of basis from linear algebra.
There is a similar to it notion in representation theory.

Let $\varphi: G \to GL_n(V)$ be a representation. Let $B = \{b_1,\dots,b_n\}$ be a basis for V, we can associate a vector space isomorphism $T: V \to \mathbb{C}^n$ by taking coordinates. Then we can define a representation $\psi: G \to GL_n(\mathbb{C})$ by $\psi_g = T\varphi_g T^{-1}$ for $g \in G$. Let $B' = \{b_1',\dots,b_n'\}$ be another basis, then we have another isomorphism $S: V \to \mathbb{C}^n$ $\Rightarrow$ we obtain another rep. $\psi': G \to GL_n(\mathbb{C})$ by $\psi_g' = S\varphi_g S^{-1}$. ψ and ψ' are related by the formula $\psi_g' = ST^{-1}\psi_g T S^{-1} = (ST^{-1})\psi_g(ST^{-1})^{-1}$.
So, φ, ψ and ψ' are the same representation.

<u>def:</u> Given are two representations $\varphi: G \to GL_n(V)$ and $\psi: G \to GL_n(W)$, $\varphi \sim \psi$ if there exists an isomorphism $T: V \to W$ s.t. $\psi_g = T\varphi_g T^{-1}$ for all $g \in G$, i.e., $\psi_g T = T\varphi_g$ for all $g \in G$.
Expressed as a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi_g"'] & W. \end{tikzcd}$$

Examples of equivalent representations:

1) $\varphi: \mathbb{Z}/n\mathbb{Z} \to GL_2(\mathbb{C})$

$$\varphi_{[m]} = \begin{bmatrix} \cos\left(\frac{2\pi m}{n}\right) & -\sin\left(\frac{2\pi m}{n}\right) \\ \sin\left(\frac{2\pi m}{n}\right) & \cos\left(\frac{2\pi m}{n}\right) \end{bmatrix},$$

rotation matrix by $\frac{2\pi m}{n}$

Recall: $\cos\theta + i\cdot\sin\theta = e^{i\theta}$

$\psi: \mathbb{Z}/n\mathbb{Z} \to GL_2(\mathbb{C})$

$$\psi_{[m]} = \begin{bmatrix} e^{\frac{2\pi m i}{n}} & 0 \\ 0 & e^{-\frac{2\pi m i}{n}} \end{bmatrix},$$

T is given by $\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$,

and its inverse $T^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}$.

One sees that φ and ψ are equivalent. Indeed,

$$T^{-1}\varphi_{[m]}T = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}\begin{bmatrix} \cos\left(\frac{2\pi m}{n}\right) & -\sin\left(\frac{2\pi m}{n}\right) \\ \sin\left(\frac{2\pi m}{n}\right) & \cos\left(\frac{2\pi m}{n}\right) \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} =$$

$$= \frac{1}{2i}\begin{bmatrix} e^{\frac{2\pi m i}{n}} & ie^{\frac{2\pi m i}{n}} \\ -e^{-\frac{2\pi m i}{n}} & ie^{-\frac{2\pi m i}{n}} \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} 2ie^{\frac{2\pi m i}{n}} & 0 \\ 0 & 2ie^{-\frac{2\pi m i}{n}} \end{bmatrix} = \psi_{[m]}.$$

2) Consider the Klein's four group $K_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ($\ncong D_2$), which has the presentation

$K_4 = \langle a, b \mid a^2 = b^2 = e,\ ab = ba \rangle$

$K_4 = \{e, a, b, ab\}$

There is a representation: $a \mapsto A = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$

$b \mapsto B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$

(verify that it satisfies the relations!)

One can find a matrix $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, s.t. A can be diagonalized, i.e. $P^{-1}AP = \tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$

Same for B, get $\tilde{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, we get an equivalent representation.

Rem: In this case it is easier to work with the new rep., since the matrices are diagonal!

The next example is very important:

<u>Exp:</u> $S_3 = \{e, (12), (23), (13), (123), (132)\}$

- Besides the trivial representation, there exists one more 1-dim. representation of S_3 (and of S_n), namely the sign representation: (alternating)

$$\varphi_\sigma v = \operatorname{sgn}(\sigma) v, \text{ where } \operatorname{sgn}(\sigma) \text{ is the sign of } \sigma \in S_3$$

$$\operatorname{sgn}(\sigma) \mapsto \begin{cases} 1, & \text{if } \# \text{transpositions is even} \\ -1, & \text{if } \quad -\text{"}- \quad \text{odd.} \end{cases}$$

- Consider the standard representation of S_n, given by $\varphi: S_n \to GL_n(\mathbb{C})$, as $\varphi_\sigma(e_i) = e_{\sigma(i)}$, where e_i is a vector of the standard basis.

For instance, if $n=3$, we have:

$$\varphi_e = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \quad \varphi_{(12)} = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}, \quad \varphi_{(23)} = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \quad \varphi_{(13)} = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix},$$

$$\varphi_{(123)} = (12)\circ(13) = (12).\begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}, \quad \varphi_{(132)} = (13).(12) = (13).\begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}.$$

<u>Remark:</u> Consider the following observation: (for S_n in general)

$$\varphi_\sigma(e_1+e_2+\dots+e_n) = e_{\sigma(1)} + e_{\sigma(2)} + \dots + e_{\sigma(n)} = e_1+e_2+\dots+e_n.$$

This means that there is a 1-dim. subspace, generated by the vector $(1,1,\dots,1)^T$. $\mathbb{C}(e_1+\dots+e_n)$

This is an example of the so called G-invariant space.

<u>def:</u> Let $\varphi: G \to GL(V)$ be a rep. A subspace $W \leq V$ is G-invariant, if for all $g \in G$ and $w \in W$, one has $\varphi_g w \in W$.

<u>Exp:</u> Obviously, $\mathbb{C}e_1$ and $\mathbb{C}e_2$ are $\mathbb{Z}/n\mathbb{Z}$-invariant and $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, from the example of equivalent reps.

We can construct new reps. out. of old ones by taking the direct sum of reps.

<u>def:</u> Given two representations $\varphi^{(1)}: G \to GL_n(V_1)$ and $\varphi^{(2)}: G \to GL_m(V_2)$. Their direct sum is defined via:

$$\varphi^{(1)} \oplus \varphi^{(2)}: G \to GL_{n+m}(V_1 \oplus V_2).$$

This is exactly $(\varphi^{(1)} \oplus \varphi^{(2)})_g (v_1, v_2) = (\varphi_g^{(1)}(v_1), \varphi_g^{(2)}(v_2))$.

In terms of matrices, we get:

$$(\varphi^{(1)} \oplus \varphi^{(2)})_g = \begin{bmatrix} \varphi_g^{(1)} & 0 \\ 0 & \varphi_g^{(2)} \end{bmatrix}.$$

Exp: Take $\varphi^{(1)}: \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ by $\varphi^{(1)}_{[m]} = e^{2\pi i m/n}$, $\varphi^{(2)}: \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ by $\varphi^{(2)}_{[m]} = e^{-\frac{2\pi i m}{n}}$.

Then $(\varphi^{(1)} \oplus \varphi^{(2)})_{[m]} = \begin{bmatrix} e^{\frac{2\pi i m}{n}} & 0 \\ 0 & e^{-\frac{2\pi i m}{n}} \end{bmatrix}$.

Remark: Note that for $n>1$, the representation $\varphi: G \to GL_n(\mathbb{C})$, mapping each $g \in G$ to Id_n is not the trivial rep. It is the direct sum of n copies of the trivial rep.

Every group has a presentation in terms of generators and relations.
Then a rep. φ of G is determined by its values on the set of generators.
Warning: Not any assignment of matrices to the generators gives a valid rep.!
(Indeed, it has to satisfy the relations.)

Example: Go back to the example with S_3. We already discussed that its 3-dim rep. contains a 1-dim. invariant subspace, which is spanned by $(1,1,1)^T$.
Denote this space $\mathbb{C}(1,1,1) := W$. From the general theory there exists an orthogonal complement to W, namely $W^\perp := \{(a,b,c) \mid a+b+c=0\}$. This space is 2-dimensional.
We take a basis, given by $\{\underset{e_2-e_1}{\overset{\parallel}{\alpha_1}}, \underset{e_2-e_3}{\overset{\parallel}{\alpha_2}}\}$.

Then, $(12).(e_2-e_1) = (12).e_2 - (12).e_1 = e_1 - e_2 = -\alpha_1$
$(12).(e_2-e_3) = (12).e_2 - (12).e_3 = e_1 - e_3 = \underbrace{e_1-e_2}_{-\alpha_1} + \underbrace{e_2-e_3}_{\alpha_2} = -\alpha_1 + \alpha_2$

$\Rightarrow \varphi(12) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$.

By the same idea, one gets $\varphi_{(123)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

To verify that it defines a rep, check if the relations for S_3 are satisfied.
$S_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i^2 = 1, \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j|>1, (\sigma_i\sigma_{i+1})^3 = 1 \rangle$

If $\psi: S_3 \to \mathbb{C}^\times$, $\psi_\sigma = 1$, we obtain:

$(\varphi \oplus \psi)_{(12)} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\varphi \oplus \psi)_{(123)} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Rem: $\varphi \oplus \psi \sim$ standard rep. of S_3.

def: Let $\varphi: G \to GL(V)$ be a rep. If $W \leq V$ is a G-invariant subspace, we restrict $\varphi|_W: G \to GL(W)$ by setting $(\varphi|_W)_g(w) = \varphi_g(w)$ for $w \in W$. Then $\varphi|_W$ defines a subrepresentation of φ.

If $V_1, V_2 \leq V$ are G-invariant and $V = V_1 \oplus V_2$, then $\varphi \sim \varphi|_{V_1} \oplus \varphi|_{V_2}$.

Let $B = B_1 \cup B_2$ be a basis for V.

Since V_i is G-invariant, we have $\varphi_g(B_i) \leq V_i = \mathbb{C}B_i$.

In terms of matrices:

$$[\varphi_g]_B = \begin{bmatrix} [\varphi^{(1)}]_{B_1} & 0 \\ 0 & [\varphi^{(2)}]_{B_2} \end{bmatrix} \quad \text{and so } \varphi \sim \varphi^{(1)} \oplus \varphi^{(2)}.$$

We want to obtain a notion of "unique factorization into primes" but for representations.

We already discussed that there exist analogies from group theory, linear algebra and representation theory:

Groups	"dictionary" $\rightsquigarrow$	Vector spaces	"dictionary" $\rightsquigarrow$	Representations
Subgroup		Subspace		G-invariant subspace
Simple group		One-dim. subspace		Irreducible representation
Direct product		Direct sum		Direct sum
Isomorphism		Isomorphism		Equivalence

def: A non-zero representation $\varphi: G \to GL(V)$ of a group G is said to be irreducible if the only G-invariant subspaces are V and $\{0\}$.

Exp: Every degree one rep. $\varphi: G \to \mathbb{C}^\times$ is irreducible, since $\mathbb{C}$ is 1-dim.

If $G = \{e\}$ and $\varphi: G \to GL(V)$ is a rep, then $\varphi_e = Id$. A rep. of the trivial group is irreducible $\Leftrightarrow$ it has degree one.

Exp: Recall the example about the equivalent reps.

$\mathbb{C}\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\mathbb{C}\begin{bmatrix} -i \\ 1 \end{bmatrix}$ are $\mathbb{Z}/n\mathbb{Z}$-invariant subspaces for φ;

$\mathbb{C}e_1$ and $\mathbb{C}e_2$ are inv. subspaces for ψ.

Thus, these representations are <u>not</u> irreducible.

Exp: The representation of S_3 given by $\varphi : S_3 \to GL_2(\mathbb{C})$ is irreducible.

Proof: $\dim \mathbb{C}^2 = 2$, then any proper G-invariant subspace W must be one-dimensional.

Let $v \neq 0 \in W$; so $W = \mathbb{C}v$. Let $\sigma \in S_3$. Then $\varphi_\sigma(v) = \lambda v$ for $\lambda \in \mathbb{C}$, since by the S_3-invariance of W, we must have $\varphi_\sigma(v) \in W = \mathbb{C}v$. It follows that v must be an eigenvector for all φ_σ with $\sigma \in S_3$.

Claim: Want to prove that $\varphi_{(12)}$ and $\varphi_{(123)}$ have no common eigenvector.

Proof: Indeed: Take $\varphi_{(12)} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$, $\lambda_1 = -1$ and $\lambda_2 = 1$.

Thus, we have the eigenspaces $V_{-1} = \mathbb{C}e_1$ and $V_1 = \mathbb{C}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Check if e_1 is an eigenvector of $\varphi_{(123)}$:

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \notin V_{-1} \quad \underline{\text{no}}$$

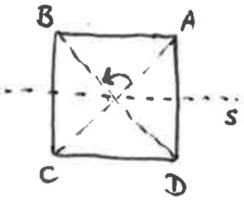
$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin V_1 \quad \underline{\text{no}}$$

$\Rightarrow$ $\varphi_{(12)}$ and $\varphi_{(123)}$ have no common eigenvector.

$\Rightarrow$ φ is irreducible. $\square$

Rmk: The trick works only for reps. of degree 2 and 3.

Exp: Consider $D_4 = \langle r, s \mid r^n = s^2 = e,\ srs = r^{-1} \rangle$



$r :=$ rotation by $\frac{\pi}{2}$

$s :=$ reflection

We have a 2-dim. rep, given by:

$$\varphi(r^k) = \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix}, \quad \varphi(sr^k) = \begin{bmatrix} 0 & (-i)^k \\ i^k & 0 \end{bmatrix}.$$

φ is irreducible

Same trick as above!

"Factorization into primes" $\Leftrightarrow$ equivalence to a direct sum of irreducible reps.

Rmk: Irreducibles are called simples sometimes.

def: Let G be a group. A representation $\varphi: G \to GL(V)$ is said to be completely reducible, if $V = V_1 \oplus V_2 \oplus \dots \oplus V_n$, where V_i are G-invariant subspaces and $\varphi|_{V_i}$ is irreducible for all $i = 1, \dots, n$.

$\Leftrightarrow$ $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)} \oplus \dots \oplus \varphi^{(n)}$, where $\varphi^{(i)}$ are irreducible representations.

def: A non-zero representation φ of a group G is decomposable if $V = V_1 \oplus V_2$ with V_1, V_2 non-zero G-invariant subspaces. Otherwise, V is called indecomposable.

$\boxed{\text{Rep. Th.}}$ Complete reducibility $\leftrightsquigarrow$ $\boxed{\text{Lin. Alg.}}$ diagonalizability

An obvious lemma:

Lemma: Let $\varphi: G \to GL(V)$ be equivalent to a decomposable/irreducible/completely reducible representation. Then φ is decomposable/irreducible/completely reducible.

Take away:

1. Equivalent reps. are "the same";
2. To know the irreducibles is good for us;
3. Irreducible $\Rightarrow$ Indecomposable
   $\cancel{\Leftarrow}$