Prologue: A vocabulary from category theory

24.09.2018

0. Motivation for TQFT

Physics
- A. Schwarz (1977) involving path integral
- E. Witten (1988)

Mathematics
- M. Atiyah (1989) → simple axiomatic description, inspired by 2d CFT of Segal
- V. Turaev
- L. Abrams (1996) 2Cob via generators and relations
- + many others

TQFT appears in
- quantum Hall effect
- quantum computing
- mirror symmetry
- invariants of 3-mnf.
- 4d topology, Donaldson
- Geometric Langlands Program

Convention: The composition of functions (arrows) is given from left to right:

i.e $X \xrightarrow{f} Y \xrightarrow{g} Z$ reads as $f \circ g : X \to Z$

Reason: Glueing along the boundary in $\mathcal{C}ob$ is more natural.

1. Categories and functors

def: A category $\mathcal{C}$ consists of:
- a class of objects $Ob(\mathcal{C})$ containing $X, Y, Z, \ldots$ objects
- a set of morphisms $\mathcal{C}(X,Y)$ for any two objects X, Y, called arrows
- there is an associative composition law: for any triple of objects $X, Y, Z \in Ob(\mathcal{C})$

we define a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$, which is associative, i.e

$$W \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z : \quad (e \circ f) \circ g = e \circ (f \circ g)$$

- for each object $X \in Ob(\mathcal{C})$, there is an identity arrow $Id_X \in \mathcal{C}(X,X)$ which acts as a neutral element for the composition, i.e $1_X \circ f = f \circ 1_Y = f$ for $f : X \to Y$

$\mathcal{C}^{op}$ is the opposite category of $\mathcal{C}$ with the same objects but reversed arrows.

Examples:

Type: objects are sets
+ additional structure,
the arrows preserve this
structure
concrete categories

	Set	Fin Ord	Grp	Ring	Top	Vect$_{\mathbb{K}}$
Objects	all sets	finite ordered sets	groups	rings	top. spaces	vector spaces over $\mathbb{K}$
Morphisms	set functions	order-preserving functions	group homom.	ring homom.	continuous maps	$\mathbb{K}$-linear maps

Type: "abstract"

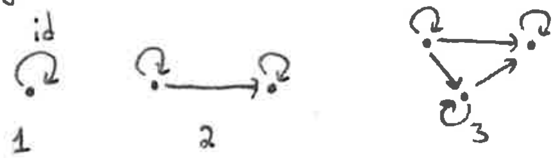
Let G be a group (more generally a monoid).
We can associate to G a category BG in the following way:

$Obj(BG) = \{*\}$ one single object

$BG(*,*) = G$

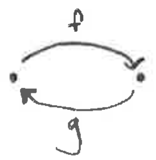
Any $g \in G$ can be regarded as an arrow $* \xrightarrow{g} *$, the neutral element is the identity arrow of $*$. The composition is given by multiplication in G. Any arrow in BG admits an inverse. Namely, the inverse of g is given by $g^{-1}: * \to *$

Type: Finite graphs

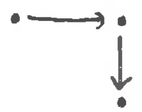


(usually we don't draw the identity arrows)

Non-examples:



no relation!
not a finite
category



no composition!
not a category

Functors encapsulate the notion of an arrow between categories.

<u>def:</u> A functor between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of:

- A map $Ob(\mathscr{C}) \to Ob(\mathscr{D})$
  $X \mapsto F(X)$

- For each pair of objects $X, Y \in Ob(\mathscr{C})$ a map $F_{X,Y}: \mathscr{C}(X,Y) \to \mathscr{D}(F(X), F(Y))$,
  $f \mapsto F(f)$

preserving the composition law and identity arrows, i.e $F(id_X) = id_{F(X)}$, $\forall X \in Ob(\mathscr{C})$ and $F(f' \circ f) = F(f') \circ F(f)$, $\forall f', f$ composible in $\mathscr{C}$.

<u>Examples:</u>

Type: forgetful functors
"forget" the structure

$Vect_{\mathbb{K}} \to Set$
$Top \to Set$
$Grp \to Set$
$FinOrd \to Set$
$Ring \to Set$

Type: free functors

① $F: Sets \to Grp$
$S \mapsto F(S)$ free group on S

$f: S \to S' \mapsto \underset{\text{grp.hom.}}{F(S) \to F(S')}$

② If $\mathbb{K}$ is a field, $F: Sets \to Vect_{\mathbb{K}}$ assigns to S the $\mathbb{K}$-v.space generated by S

$F(S) = \{ f: S \to \mathbb{K} \mid \{ s \in S : f(s) \neq 0 \} \text{ is finite} \}$ has the structure of a vector space.

One can think of $v \in F(S)$ as $\sum_{s \in S} v(s).s$ ← formal lin. combinations

Addition: $(f+g)(s) = f(s) + g(s)$

mult. in $\mathbb{K}$

Scalar multiplication: $(\lambda f)(s) = \lambda \cdot f(s)$

"Fancy" examples:

① $Euclid_*$ category with objects pointed Euclidean spaces $(\mathbb{R}^n, a)$ and arrows are differentiable functions $f: \mathbb{R}^n \to \mathbb{R}^m$ s.t. $f(a) = b$.

We have a functor $D: Euclid_* \to Mat(\mathbb{R})$ defined as follows:

Ob: $(\mathbb{R}^n, a) \to D(\mathbb{R}^n, a) = n$, the dimension

Mor: $f: (\mathbb{R}^n, a) \to (\mathbb{R}^m, b) \mapsto \left\{ \frac{\partial f_i}{\partial x_j} \Big|_a \right\} =: J_a(f)$ ← Jacobian evaluated in a

D preserves the composition, since the chain rule holds true.

② G group

Consider a functor $F: BG \to Set$.

A set $S := F(*)$ ← obj. of BG

For any element $g \in G$ we have $F(g): S \to S$ s.t. $F(e) = id_S$ and $F(gg') = F(g) \circ F(g')$.

We have a function $G \times S \to S$
$(g, s) \mapsto F(g)(s) = g \cdot s$ s.t. $(g'g) \cdot s = g' \cdot (g \cdot s)$ $\forall s \in S$ and $\forall g, g' \in G$

This is the definition of a <u>left G-set</u>.

For $F: BG \to Vect_{\mathbb{K}}$ we get the definition of a <u>G-representation</u>.

<u>def</u>: A contravariant functor is a functor $\mathcal{C}^{op} \to \mathcal{D}$, which reverses the directions of all arrows.

<u>Examples</u>:

$F: Vect^{op} \to Vect$
$V \mapsto V^*$

$F: Top^{op} \to Ring$
$X \mapsto C(X)$ ← ring of all cont. functions $X \to \mathbb{R}$

$X \xrightarrow{f} Y \mapsto C(Y) \to C(X)$ given by sending $Y \to \mathbb{R}$ to the composite $X \to Y \to \mathbb{R}$

There always exists the identity functor $Id_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$.
We can compose functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to E$ to get a functor $F \circ G: \mathcal{C} \to E$, thus we can define a functor category $[\mathcal{C}, \mathcal{D}]$.

def: A functor is an isomorphism of categories, if there is a functor in the other direction, s.t. the two compositions are both equal to the identity functors.

$F: \mathscr{C} \to \mathscr{D}$ and $G: \mathscr{D} \to \mathscr{C}$ s.t. $F \circ G = id_{\mathscr{C}}$ and $G \circ F = id_{\mathscr{D}}$.

def: A functor $F: \mathscr{C} \to \mathscr{D}$ is called:

- faithful, if for each pair of objects $X, Y \in Ob(\mathscr{C})$ the map $F_{X,Y}: \mathscr{C}(X,Y) \to \mathscr{D}(F(X), F(Y))$ is injective;
- full, if the maps $F_{X,Y}: \mathscr{C}(X,Y) \to \mathscr{D}(F(X), F(Y))$ are all surjective;
- essentially surjective, if every object in $\mathscr{D}$ is isomorphic to an image under F of an object of $\mathscr{C}$.

(Isomorphism in cat. theory means that for an arrow $f: X \to Y$ $\exists$ a morphism $g: Y \to X$ s.t. $g \circ f = id_Y$ and $f \circ g = id_X$)

2. Natural transformations

"morphisms between functors"

def: $\mathscr{C}$ and $\mathscr{D}$ two categories

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{D} \end{tikzcd}$$

a nat. transformation $\alpha: F \Rightarrow G$ is the data of:

- For each object $X \in Ob(\mathscr{C})$ and $\alpha_X: F(X) \to G(X)$ in $\mathscr{D}$, these maps must be natural; i.e for $f: X \to Y$ in $\mathscr{C}$, this diagram must commute:

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"'] & G(Y) \end{tikzcd}$$

We have identity nat. transformation $\alpha: F \Rightarrow F$, $\circ$ is associative. The nat. transformations are the morphisms in the functor categories.

def: A functor F is an equivalence, if it is faithful, full and essentially surjective.

$\Updownarrow$ equivalently

$F \circ G: \mathscr{C} \to \mathscr{C}$ is a nat. isomorphism to $Id_{\mathscr{C}}$ and $G \circ F: \mathscr{D} \to \mathscr{D}$ is a nat. isomorphism to $Id_{\mathscr{D}}$.

an isomorphism in $[\mathscr{C}, \mathscr{D}]$

Examples:

① $(-)^{**}: \mathrm{Vect}_{\mathbb{K}} \to \mathrm{Vect}_{\mathbb{K}}$ of any dimension

$V \mapsto (V^*)^*$

$f \mapsto (f^*)^*$

$\exists$ a lin. map $\alpha_V : V \to V^{**}$
$v \mapsto \alpha_V(v) : \underbrace{V^* \to \mathbb{K}}^{ev_v}$
$\varphi \mapsto \varphi(v)$

$$\begin{array}{ccc} V & \xrightarrow{ev_V} & V^{**} \\ \phi \downarrow & & \downarrow \phi^{**} \\ W & \xrightarrow[ev_W]{} & W^{**} \end{array}$$

$\alpha = \{\alpha_V\}_{V \in \mathrm{Vect}_{\mathbb{K}}}$ family of arrows

$\alpha : \mathbb{1}_{\mathrm{Vect}_{\mathbb{K}}} \Rightarrow (-)^{**}$ nat. transf.

$f: X \to Y \mapsto F(f): F(X) \to F(Y)$

② $\alpha : \mathbb{1}_{\mathrm{Vect}^{f.d}_{\mathbb{K}}} \cancel{\Rightarrow} (-)^*$, because $\mathrm{Id}_{\mathrm{Vect}_{\mathbb{K}}}$ is covariant and $(-)^*$ is contravariant.

③ G a group

$X : BG \to \mathcal{C}$ functor corresponding to an object $X \in \mathcal{C}$ equipped with a left action of G

$X, Y : BG \to \mathcal{C}$

Because BG has only one object, the data of $\alpha : X \Rightarrow Y$ consists of a single morphism $\alpha : X \to Y$ in $\mathcal{C}$ that is G-equivariant, meaning that for each $g \in G$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ g_* \downarrow & & \downarrow g_* \\ X & \xrightarrow[\alpha]{} & Y \end{array}$$

commutes.

3. Universal objects (or abstract structures in categories)

def: Let $\mathcal{C}$ be a category. An object $I \in Ob(\mathcal{C})$ is called initial, if $\forall X \in Ob(\mathcal{C})$ $\exists!$ $f \in \mathcal{C}(I, X)$.

Dually, take an object $T \in Ob(\mathcal{C})$, s.t. $\forall X \in Ob(\mathcal{C})$, there is precisely one arrow $X \to T$.

Rmk: Initial and terminal objects are unique up to an isomorphism (if they exist).
The terminal object in $\mathcal{C}$ is just an initial object in $\mathcal{C}^{op}$!

Examples:

category of all small categories

	Set	Cat	Grp	Ring	$\mathrm{Vect}_{\mathbb{K}}$
initial object	$\emptyset$	0 (empty cat)	$\{e\}$	$\mathbb{Z}$	$\{0\}$
terminal object	$\{x\}$	1	$\{e\}$	$\{x\}$	$\{0\}$

Non-example: BG does not have an initial or terminal object, unless G is trivial.

$$\begin{tikzcd} \bullet \arrow[r] & * \\ * \arrow[u, bend left] \arrow[u, bend right] \arrow[ur] & \end{tikzcd}$$

no initial object

def: $\mathcal{C}$ is a category, X and Y are two objects. A categorical product of X and Y is an object P together with two arrows $p_1 : P \to X$, $p_2 : P \to Y$ called projections, with the following universal property: for every other object Q equipped with arrows $X \leftarrow Q \to Y$ there is a unique arrow $Q \to P$ which makes the diagram commute:

$$\begin{tikzcd} & Q \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

Rmk: If the product exists, it is unique up to a unique isomorphism.

Examples: $\mathcal{S}et$, $Vect_{\mathbb{K}}$, Top, $\mathcal{C}at$ have the cartesian product

$n\mathcal{C}ob$ has no cat. product

def: A coproduct of X and Y is an object C with two arrows $X \to C \leftarrow Y$, which makes the diagram commute:

$$\begin{tikzcd} & D & \\ & C \arrow[u, dashed, "\exists!"] & \\ X \arrow[ur, "i_1"] \arrow[uur, bend left, "f"] & & Y \arrow[ul, "i_2"'] \arrow[uul, bend right, "g"'] \end{tikzcd}$$

Examples: In $\mathcal{S}et$, Top, $\mathcal{C}at$ we have $\amalg$ (disjoint union) as a coproduct.

For $Vect_{\mathbb{K}}$ it is $\oplus$ (direct sum).

In a similar way we can define the n-ary products:

$$\begin{tikzcd} & & Q \arrow[d, dashed, "\exists!"] \arrow[dll, bend right] \arrow[dl, bend right] \arrow[d] \arrow[drr, bend left] & & \\ & & P \arrow[dll] \arrow[dl] \arrow[drr] & & \\ \cdots \; X_i & X_{i+1} & & & X_{i+2} \; \cdots \end{tikzcd}$$

def: Let $\mathcal{C}$ be a category. Consider the following diagram:

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

A pullback (or fibre product) P for this diagram is an object in $\mathcal{C}$ together with arrows $p_1: P \to X$ and $p_2: P \to Y$ s.t. the following diagram is commutative:

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

and such that the following universal property holds: for any object $A \in \mathcal{C}$ and any $f: A \to X$, $g: A \to Y$ s.t. the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

$\exists!\ \bar{f}: A \to P$ s.t. the diagram

$$\begin{tikzcd} A \arrow[dr, dashed, "\bar{f}"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd}$$

commutes, i.e

we have that $f = \bar{f} \circ p_1$
and $g = \bar{f} \circ p_2$

Example:
Consider the following diagram in $\mathcal{S}et$, take the set P defined as $P := \{(x,y) \in X \times Y \mid f(x) = g(y)\}$

$$\begin{tikzcd} P \arrow[r, dashed, "p_2"] \arrow[d, dashed, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

$p_1: P \to X$
$(x,y) \mapsto x$

$p_2: P \to Y$
$(x,y) \mapsto y$

Reversing all the arrows in the definition of a pullback gives the definition of a pushout.

4. Monoidal categories and more

def: A monoidal category is a quintuple $(\mathcal{C}, \otimes, \alpha, 1, \iota)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, called tensor product bifunctor;
$\alpha : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ nat. isomorphism with components $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ for $\forall X, Y, Z \in Ob(\mathcal{C})$, called associativity isomorphism;
$1 \in \mathcal{C}$ unit object and $\iota : 1 \otimes 1 \xrightarrow{\sim} 1$ isomorphism, s.t. the following two axioms are satisfied:

① The pentagon axiom:

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\alpha_{W,X,Y} \otimes id_Z"'] \arrow[dr, "\alpha_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\alpha_{W, X \otimes Y, Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "id_W \otimes \alpha_{X,Y,Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

is commutative for all objects W, X, Y, Z in $\mathcal{C}$.

② The functors $L_1 : X \mapsto 1 \otimes X$
$R_1 : X \mapsto X \otimes 1$ are autoequivalences of $\mathcal{C}$.

def: A strict monoidal category is a triple $(\mathcal{C}, \otimes, 1)$, where for each triple of objects $X, Y, Z \in Ob(\mathcal{C})$ we have $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $X \otimes 1 = 1 \otimes X = X$.

Rmk: Most of the categories that we know are not strict. But we can always "strictify" them.

Maclane Strictness Thm: Any monoidal category is monoidally equivalent to a strict one.

Examples:

$(\mathcal{S}et, \times, \{*\})$

$(\mathcal{S}et, \amalg, \emptyset)$

$(Vect_{\mathbb{K}}, \otimes, \mathbb{K})$

$(2Cob, \sqcup, \emptyset)$ → to be defined later, is strict

$(Cat, \times, 1)$

Monoidal functors

<u>def</u>: Let $(\mathcal{C}, \otimes, 1, \alpha, \iota)$ and $(\mathcal{C}', \otimes', 1', \alpha', \iota')$ be two monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a pair $(F, \mathcal{J})$, where $F: \mathcal{C} \to \mathcal{C}'$ is a functor, and:

$\mathcal{J}_{X,Y}: F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y)$ is a natural isomorphism s.t. $F(1) \cong 1'$ and the diagram commutes:

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "\alpha'_{F(X),F(Y),F(Z)}"] \arrow[d, "\mathcal{J}_{X,Y} \otimes' id_{F(Z)}"'] & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "id_{F(X)} \otimes' \mathcal{J}_{Y,Z}"] \\
F(X \otimes Y) \otimes' F(Z) \arrow[d, "\mathcal{J}_{X \otimes Y, Z}"'] & F(X) \otimes' F(Y \otimes Z) \arrow[d, "\mathcal{J}_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(\alpha_{X,Y,Z})"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$\forall X, Y, Z \in \mathcal{C}$.

Similarly to strict monoidal categories, there exist strict monoidal functors.

<u>def</u>: Let $(\mathcal{C}, \otimes, 1)$ and $(\mathcal{C}, \otimes', 1')$ be strict monoidal categories. A <u>strict</u> monoidal functor ~~between $\mathcal{C}$ and $\mathcal{C}'$~~ is a functor $F: \mathcal{C} \to \mathcal{C}'$ s.t. $F(X) \otimes' F(Y) = F(X \otimes Y)$ and $F(1) = 1'$.

<u>def</u>: A symmetric monoidal category is a monoidal category $\mathcal{C}$ with a natural isomorphism $\sigma_{X,Y}: X \otimes Y \xrightarrow{\sim} Y \otimes X$ between two functors $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ s.t. $\sigma_{X,Y} \circ \sigma_{Y,X} = id_{X \otimes Y}$ and the two hexagon axioms are satisfied:

(I)

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[dl, "\sigma_{X,Y} \otimes id_Z"'] & X \otimes (Y \otimes Z) \arrow[dr, "\sigma_{X, Y \otimes Z}"] & \\
(Y \otimes X) \otimes Z \arrow[dr, "\alpha_{Y,X,Z}"'] & & & (Y \otimes Z) \otimes X \arrow[dl, "\alpha_{Y,Z,X}"] \\
 & Y \otimes (X \otimes Z) \arrow[r, "id_Y \otimes \sigma_{X,Z}"'] & Y \otimes (Z \otimes X) &
\end{tikzcd}$$

(II) similar to (I).

<u>Example</u>: • $Vect_{\mathbb{K}}$ is a symmetric monoidal category with symmetry $\sigma: V \otimes W \xrightarrow{\sim} W \otimes V$ given by swapping the elements of V and W, i.e. for $v \in V$ and $w \in W$ we have $v \otimes w \mapsto w \otimes v$.

• TQFT is a symmetric monoidal functor.

$$\boxed{TQFT: 2Cob \to Vect_{\mathbb{K}}}$$

2Cob is a symm. mon. category → to be defined in the further talks

5. Presentation via generators and relations

Motivation: group theory

Let G be a finite group. A generating set S is a subset $S \subset G$ s.t. each element of G can be written as a product of elements in S and their inverses.

A relation is the equality of two ways of writing a given element in terms of the generators.

Examples:

- $\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle$ — Notation: generators, relations

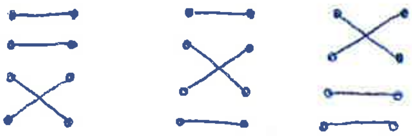
- The free group on S has no relations: $F(S) = \langle S \mid \emptyset \rangle$
- Cyclic group of order n: $C_n = \langle g \mid g^n = 1 \rangle$
- The symmetric group S_k on $k \geq 4$ letters $\{x_1, .., x_k\}$.

  S_k is generated by transpositions $\tau_i = (x_i, x_{i+1})$ for $i = 1, .., k-1$, subject to the relations: $\tau_i^2 = id$

  $$\tau_i \tau_j = \tau_j \tau_i \text{ for } j > i+1,$$
  $$\tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j \text{ for } j = i+1.$$

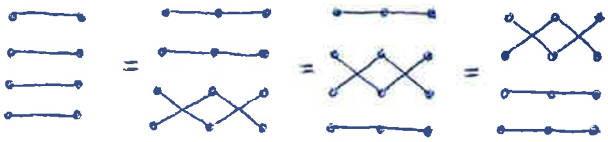
We can think of S_k as the category of invertible maps $\{x_1, .., x_k\} \to \{x_1, .., x_k\}$.
In terms of pictures:

The generators for S_4:

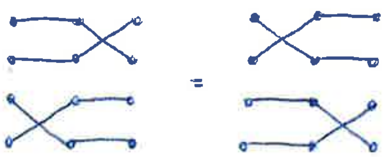


(34) (23) (12)

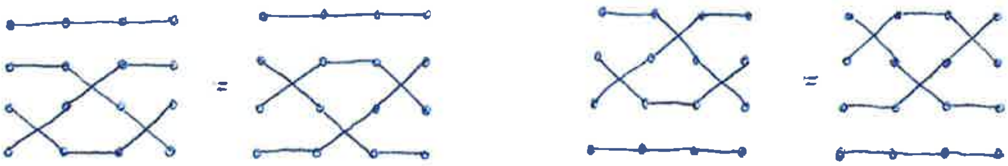
Relations:



corresponds to $\tau_i^2 = id$



corresponds to $\tau_i \tau_j = \tau_j \tau_i$ for $j > i+1$



correspond to $\tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j$ for $j = i+1$

The same idea can be applied to categories: any group can be viewed as a category with only one object BG.

def: A generating set for a category $\mathcal{C}$ is a set S of arrows s.t. every arrow in $\mathcal{C}$ can be obtained as a composition of arrows of S. A relation is the equality of two ways of writing a given arrow in terms of generators.

{too many objects} {minimal category with some property}

Remark: For large categories like $\mathrm{Vect}_{\mathbb{K}}$ or $n\mathrm{Cob}$ take a skeleton of the category. This is a full subcategory containing exactly one object of each isomorphism class.

Let $Z \subset \mathcal{C}$ be a skeleton, the embedding $Z \hookrightarrow \mathcal{C}$ is an equivalence (full, faithful and essentially surjective).

Example: Consider $\mathrm{Vect}_{\mathbb{R}}^{\text{fin. dim.}}$.

Since all vector spaces of dim n are isomorphic to $\mathbb{R}^n$, as skeleton we take the subcategory consisting of vector spaces $\mathbb{R}^n$, $n \geq 0$.

Morphisms of $\mathrm{Vect}_{\mathbb{R}}^{\text{f.d.}}$ are given by $(m \times n)$-matrices.

Every matrix M of size $m \times n$ can be written as a product (composition).

$$M = \underbrace{A}_{\substack{(m\times m)\\ \text{invertible}}} \; \underbrace{D}_{r\left(\begin{smallmatrix} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} & 0 \\ 0 & 0 \end{smallmatrix}\right)} \; \underbrace{B}_{\substack{(n\times n)\\ \text{invertible}}}$$

D: zero except for an $(r\times r)$-minor

Final remark: In this sense we can find some "elementary" cobordisms, which will be the building blocks for $2\mathcal{C}ob$.