Two-dimensional Cobordisms algebraically II

Recall from Nino's talk:

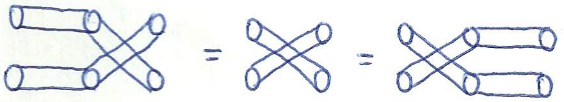
2Cob is generated as a symmetric monoidal category under composition (serial connection) and disjoint union (parallel connection) by the following six cobordisms:

"cup" "pants" "cylinder" "co-pants" "cap" "twist"

2Cob can be completely described via generators and relations.

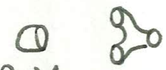
1. Relations without a twist

Identity relations: (we saw previously that the cylinders define the unit in our category)



Sewing discs relations (unit/counit)

⚠ Attaching discs to one of the holes of the pair-of-pants is not a well-defined composition of cobordisms.

 not composible, to fix this issue one has to glue cylinders.

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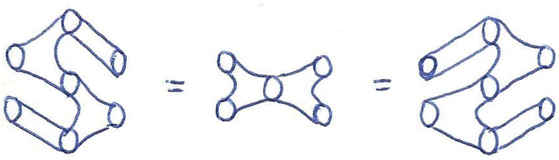
(Co-)Associativity relations

(Co-)Commutativity relations:

The Frobenius relation:

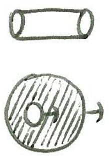


The proof of these relations is quite trivial, since we know that a cobordism class in the connected case is completely determined by the genus and the number of in- and out-boundaries.

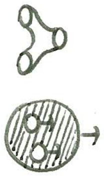
An <u>alternative proof</u> (from the view of nested discs):

Consider the left-hand side of the relations involving pants and a cup.

no in-boundary on the left, so it's a D^2



one single in-boundary disc with a missing disc



disc with two discs missing

The graphical representation is useful for the decomposition of the relations.
We prove only the left-hand side of the relations, for the right-hand side use the reverse orientation.

- Proof of the unit/counit relation:

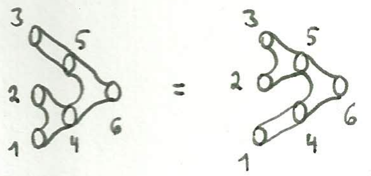
Take a cylinder and cut it along the disjoint union of two circles:

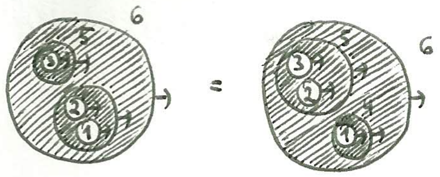
 is a composition of  with 

it is just sewing a cylinder and a disc in a pair-of-pants

- Proof of the associativity:

Number the circles to indicate the order of the glueing:



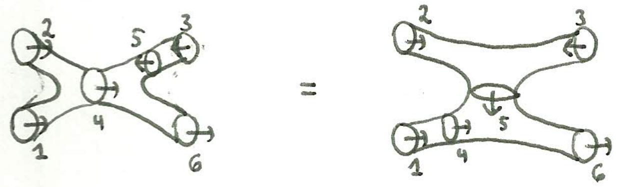


- Proof of the commutativity

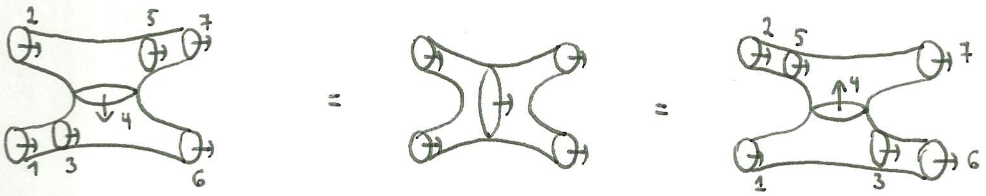
we can move the two in-boundaries around freely in 

- Proof of the Frobenius relation

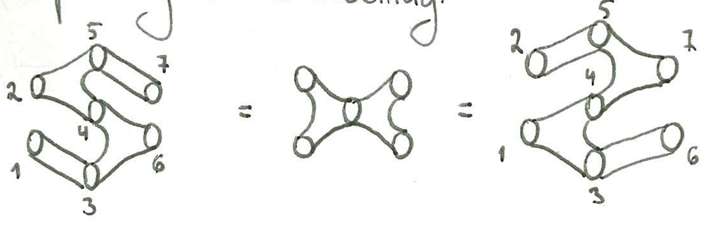
Using the labelling from the previous proof, we can draw the associativity the following way:



If we reverse the orientation of circle 3 (and change it to 7), we get a surface that can be cut in these three ways:

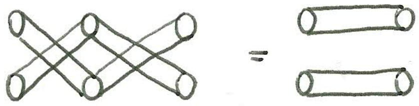


corresponding to the labelling:



2. Relations involving the twist

Notice that the twist is its own inverse:

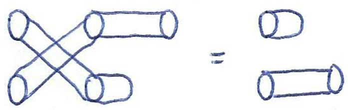


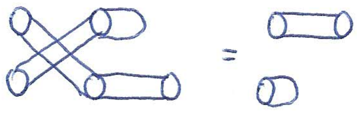
id: 2→2

$\sigma^2 = id$ → symmetric monoidal structure (Talk 1)

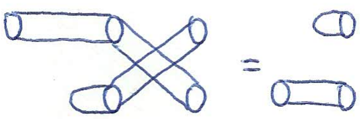
Naturality means that it does not matter whether we apply the twist before the disjoint union of two cobordism or after.

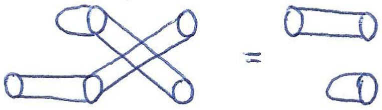
Twist with a cap



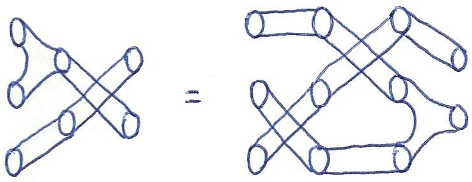


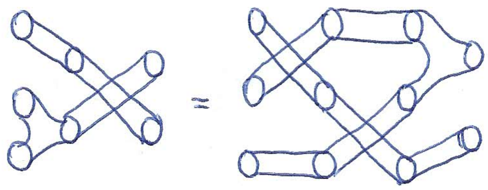
Twist with a cup

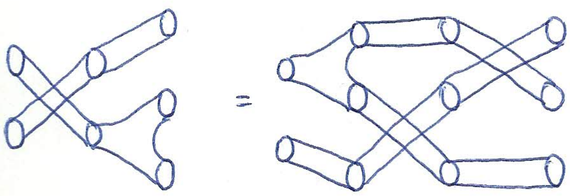


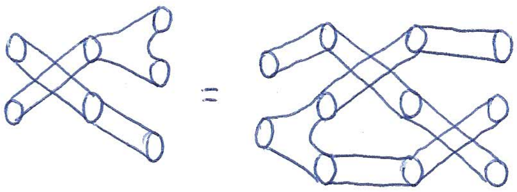


Twist involving pants and co-pants

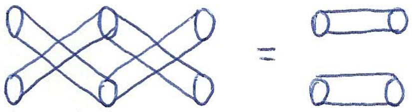




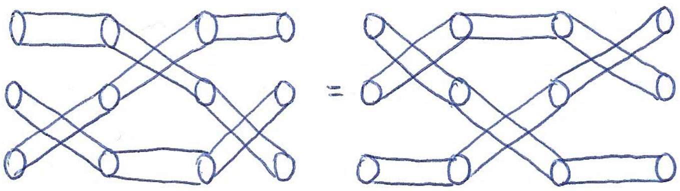




Symmetry relation



Yang-Baxter (or Braid relation)



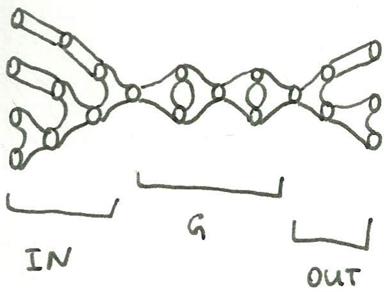
3. Sufficiency of the relations

A natural question that arises is: are these relations enough to span 2Cob?

In general it is difficult to show such a completeness. We have to work with normal form and check that the listed relations are sufficient to transform any general expression into a normal form.

Recall from the last talk what a normal form is:

Consider the example $M: \underbrace{4}_{m} \to \underbrace{3}_{n}$, a cobordism of genus 2.
The normal form of M looks like:



$M = IN \circ G \circ OUT$

Here $m=4$, $n=3$, $g=2$

For simplicity, we first observe the case when we don't have a twist, when we bring M to a normal form and later we'll observe what happens when a twist occurs. Both cases are for connected surfaces.

Introduce sort of an "algorithm" how to bring a cobordism class into its normal form.

STEP 1: "Counting pieces"

Assume that $M: m \to n$ is a connected cobordism of genus g.
Its Euler characteristic is $\chi(M) = 2-2g-m-n$.
Assume that M consists of a pants, b co-pants, p cups and q caps.

$\chi(\ ) = 1 = \chi(\ )$

$\chi(\ ) = -1 = \chi(\ )$, use the fact that χ is additive $\Rightarrow \chi(M) = p+q-a-b$

we get the equation $2-2g-m-n = p+q-a-b$ (1)

For the number of the in- and out-boundaries we get: $a+q+n = b+p+m$ (2)

From (1) and (2) follows that $a = p+m+g-1$

$b = q+n+g-1$

STEP 2: "Moving to the left"

We want to move m-1 copies of to the left until they come before any .

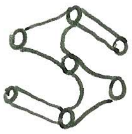
We can meet:
- a cylinder, it's just an identity, so ignore
- could be a cup, by the identity relation get a cylinder, so remove;
  can happen p times, we're left with $m-1+g$ copies of pants.
- can't meet a cap () due to connectivity.

We can meet a pair of co-pants, that could occure in two ways:

(i) 

"handle"

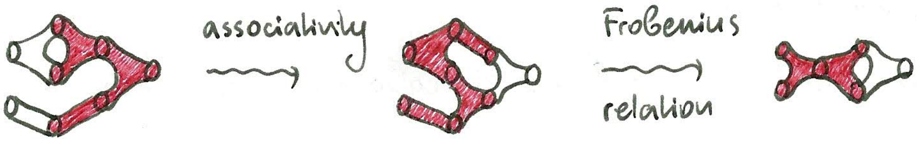
or

(ii) 

For (i) we don't have a relation, so we leave it in this form. This means we can produce the genus part, since we get g times a handle.

For (ii) we have the Frobenius relation.

<u>Remark:</u> If the handle occurs in the IN-part, we can move it to the genus part by the following relation:



We're left with $m-1$ copies of , since from the initial $m+g-1+p$, p copies met and vanished, g formed handles.

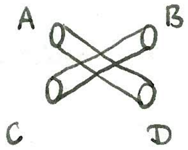
We can do a similar thing and move $n-1$ copies of the co-pants to the right to form the OUT-part.

The middle part has the same number of pants and co-pants.
The left-most is , take a and move it left to form a handle.
Continue until we get a chain of handles, hence the G-part is done.

Examine the case if we have twists.

By induction consider only one twist T.

Let T be the twist in the decomposition:



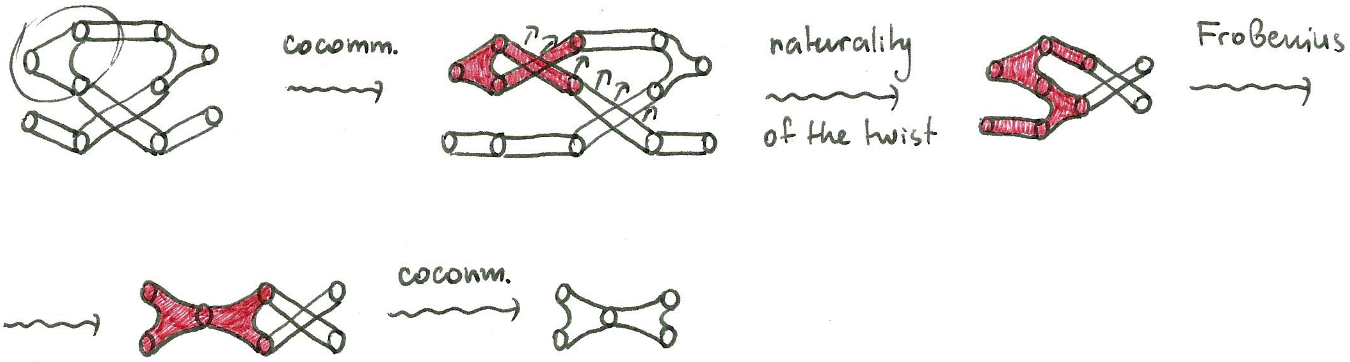
There are other pieces parallel with T but we can always insert cylinders.

The surface is connected, so some of the regions A, B, C, D must be connected with each other. This can happen in 4 possible ways.

- Suppose that A and C are connected, they form a connected surface that can be brought to a normal form by using the relations.
  In this case only the outpart of this surface will be connected with the twist. We can shuffle the co-pants up and down until we obtain a piece of the following form:

  For B and D connected, do the same.

- If A and B are connected, we have the following situation:



Same for C and D connected.

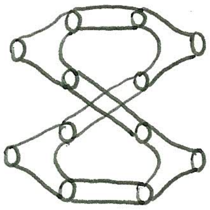
The case of <u>non-connected</u> surfaces:

We need a normal form for such surfaces.
Start with any 2-cobordism M built up of the six generators.
We know (from the last talk) that there is a pair of permutation cobordisms S and T, s.t. SMT is a disjoint union of connected components.
Notice that each of the S^{-1}, S, T and T^{-1} can be built up from twist cobordisms.
Inserting $S^{-1}S$ and TT^{-1} leads to $M = S^{-1}SMTT^{-1}$ that can be achieved by the Yang-Baxter relation and the symmetry relation.
Each of the connected components in the middle piece SMT can be brought on normal form using the idea for the connected surfaces.

<u>Remark</u>: The normal form is not unique, but any two normal forms differ only be permutations but we have the twist relations which are sufficient to realise any permutation.

Exercise 5/p.77

Show that



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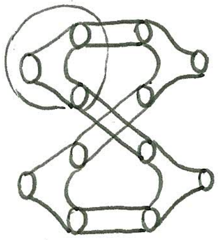
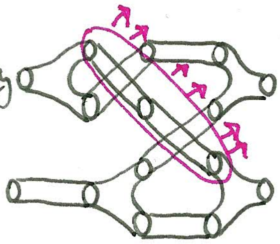
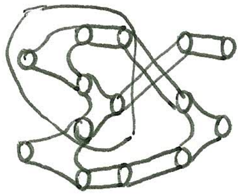


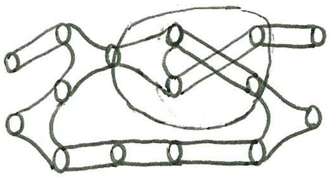
cocommutativity



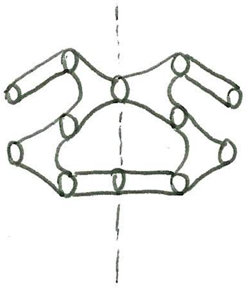
naturality of the twist



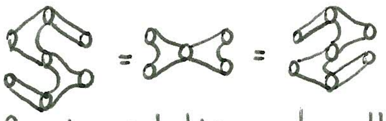
Frobenius relation + add cylinders



cocommutativity

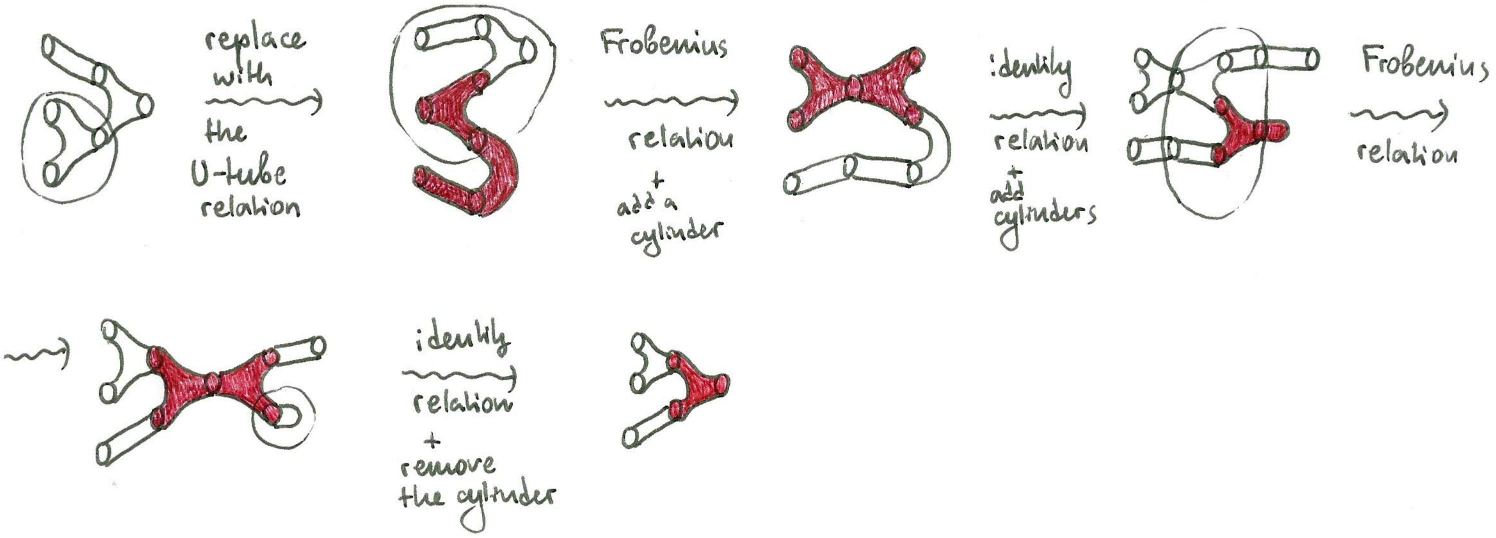


Frobenius left and right

Exercise 6 / p. 77 

To prove: The Frobenius relation together with the unit/counit relations imply the associativity and coassociativity.

We can define an "extra" relation involving the U-tube:



Similarly, for the coassociativity one can define a new U-tube relation and use it to prove the claim.