The Diagrammatic Theory Part I

Katja Warth

Universität Zürich

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You will see:

- Linear diagrams
- String diagrams
- Drawing monoidal categories
- The Temperley-Lieb category
- Another example of monoidal categories
- More isotopy (in $\mathcal{C}$at)
- Examples!

Linear diagrams

Linear diagrams

Setting:

- Category C
- Objects M, N, P,...
- morphims $f \in \mathrm{Hom}_C(M, N)$, $g \in \mathrm{Hom}_C(N, P)$, ...

linear diagrams

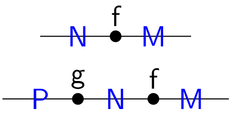
How do we draw diagrams of categories?

„classical“ diagrams:

$$M \xrightarrow{f} N$$

$$M \xrightarrow{f} N \xrightarrow{g} P$$

linear diagrams:

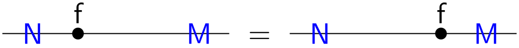


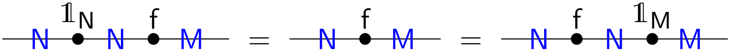
Attention:

We read linear diagrams from right to left!

linear diagrams

Remark

(1) 

(2) 

Proposition:

A diagram represents a morphism unambiguously up to linear isotopy i.e. we can stretch intervals and slide dots along a line, as long as we don't slide them past other dots.

String diagrams

string diagrams

Definition:

A (strict) 2-category consists of:

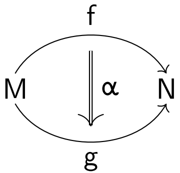
- objects N, M, ...
- 1-morphisms f, g, ... from objects to objects with assoziative composition
- 2-morphisms α, β, ... from morphisms to morphisms with vertical and horizontal composition

Example:

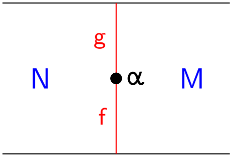
The category $\mathcal{C}$at with categories as objects, functors as 1-morphisms and natural transformations as 2-morphisms is a 2-category.

string diagrams

„classical“ diagrams:



string diagrams:



Notice:
objects: 0-dimensional
1-morphisms: 1-dimensional
2-morphisms: 2-dimensional

Notice:
objects: 2-dimensional
1-morphisms: 1-dimensional
2-morphisms: 0-dimensional

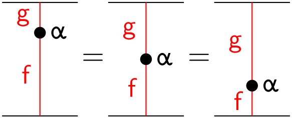
String diagrams - Examples

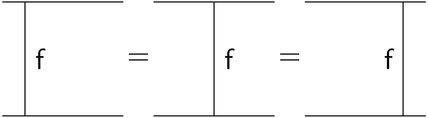
1. $\alpha = \mathrm{id}_f : f \to f$

Notice: Emptyness is identity!

2. $\alpha = \mathrm{id}_{\mathbb{1}_C} : \mathbb{1}_C \to \mathbb{1}_C$

3. Identities



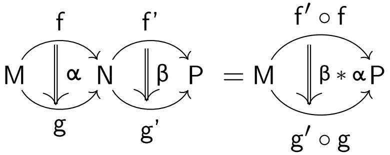


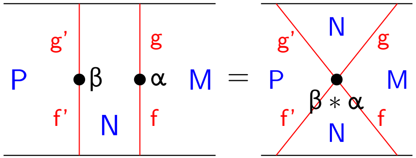
Proposition:

The axioms of a (strict) 2-category imply that a string diagram unambiguously represents a 2-morphism up to rectilinear isotopy.

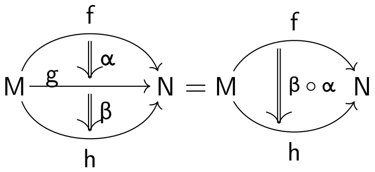
String diagrams - Examples

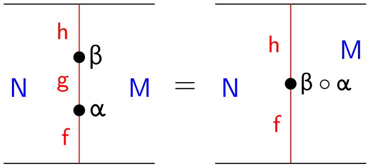
4. horizontal composition





5. vertical composition





Drawing monoidal categories

Drawing monoidal categories

Definition:

A (strict) **monoidal category** is a category equipped with an assoziative tensor product, giving us tensor objects $M \otimes N$ and tensor morphisms $f \otimes g$.

We can view a monoidal category $\mathcal{C}$ as a 2-category with a single object. Objects in $\mathcal{C}$ become 1-morphisms and 1-morphisms in $\mathcal{C}$ become 2-morphisms. The tensor product gives the composition.

Drawing monoidal categories - Example

Definition:

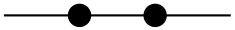
The **symmetric category** is a $\mathbb{C}$-linear monoidal category with:

- **objects:** natural numbers with tensor product $m \otimes n = m + n$.
- **morphisms:** Hom(m,n)$= \delta_{m,n}\mathbb{C}[S_n]$ with tensor product $f \otimes g =$ inclusion of group algebras $\mathbb{C}[S_n] \times \mathbb{C}[S_m] \hookrightarrow \mathbb{C}[S_{n+m}]$.

Drawing monoidal categories - Example

objects: generating object: 

monoidal identity:

tensor product: concatenation i.e. $2 =$ 

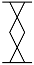
morphisms: identity morphism $\in \mathbb{C}[S_1]$: 

transposition $s \in \mathbb{C}[S_2]$: 

tensor product: horizontal concatenation

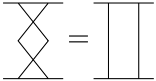
e.g. $1 \otimes s =$ 

composition: vertical concatenation

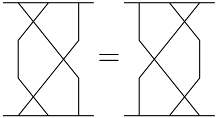
e.g. $s \circ s =$ 

Drawing monoidal categories - Example

relations: quadratic relation



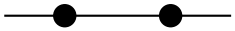
braid relation



The Temperley-Lieb Category

Some definitions

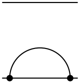
Consider $\mathrm{Vect}_{\mathbb{C}}$ with usual tensor product.

- $V := \mathbb{C}^2 = \mathbb{C}\langle e_1, e_2 \rangle$
  We draw:
  $V =$ 
  monoidal identity $\mathbb{C}$:
  tensor product $V \otimes V$: 

- Function n:

$$n : V \otimes V \to \mathbb{C}$$
$$e_1 \otimes e_1 \mapsto 0$$
$$e_2 \otimes e_2 \mapsto 0$$
$$e_1 \otimes e_2 \mapsto -1$$
$$-e_2 \otimes e_1 \mapsto -1$$

We draw:

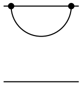
$n :=$ 

Some definitions

- Function u:

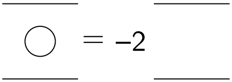
$$u : \mathbb{C} \to V \otimes V$$
$$1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$$

We draw:

u := 

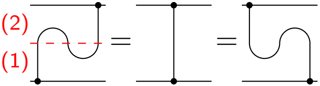
Examples

1. $n \circ u : \mathbb{C} \to \mathbb{C}$



$$(n \circ u)(1) = n(e_1 \otimes e_2 - e_2 \otimes e_1) = -2$$

2. identity relations



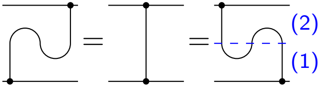
(1): $id_V \otimes u : V \otimes \mathbb{C} \to V \otimes V \otimes V$
(2): $n \otimes id_V : V \otimes V \otimes V \to \mathbb{C} \otimes V$

$$((n \otimes id_V) \circ (id_v \otimes u))(e_1 \otimes 1) = (n \otimes id_V)(e_1 \otimes (e_1 \otimes e_2 - e_2 \otimes e_1)) = (n \otimes id_V)(e_1 \otimes e_1 \otimes e_2 - e_1 \otimes e_2 \otimes e_1) = 1 \otimes e_1$$

$$((n \otimes id_V) \circ (id_v \otimes u))(e_2 \otimes 1) = (n \otimes id_V)(e_2 \otimes (e_1 \otimes e_2 - e_2 \otimes e_1)) = (n \otimes id_V)(e_2 \otimes e_1 \otimes e_2 - e_2 \otimes e_2 \otimes e_1) = 1 \otimes e_2$$

Examples

2. identity relations

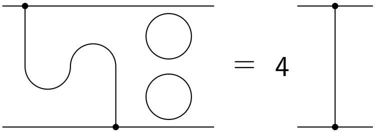


(1): $u \otimes id_V : \mathbb{C} \otimes V \to V \otimes V \otimes V$
(2): $id_V \otimes n : V \otimes V \otimes V \to V \otimes \mathbb{C}$

$$((id_V \otimes n) \circ (u \otimes id_V))(1 \otimes e_1) = (id_V \otimes n)((e_1 \otimes e_2 - e_2 \otimes e_1) \otimes e_1) = (id_V \otimes n)(e_1 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1) = e_1 \otimes 1$$

$$((id_V \otimes n) \circ (u \otimes id_V))(1 \otimes e_2) = (id_V \otimes n)((e_1 \otimes e_2 - e_2 \otimes e_1) \otimes e_2) = (id_V \otimes n)(e_1 \otimes e_2 \otimes e_2 - e_2 \otimes e_1 \otimes e_2) = e_2 \otimes 1$$

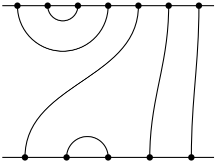
3. Simplification



Crossingless matchings

Notice that our choice of notation leads to not only rectilinear isotopy, but rather true isotopy. Using topological arguments we see that all diagrams made from cups, caps and identity morphisms are spanned by so called **crossingless matchings**.

Example:



Remark:

The above diagram represents a morphism from $V^{\otimes 5}$ to $V^{\otimes 7}$, i.e. a $2^7 \times 2^5$ matrix. It would be quite time consuming to do a composition via matrix multiplication, while the composition of diagrams is easily done, even by hand. We see: Diagrammatic computations are extremely efficient!

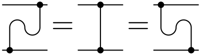
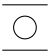
The Temperley-Lieb category

Definition:

The Temperley-Lieb category $\mathcal{TL}$ is is the $\mathbb{C}$-linear monoidal category given by:

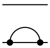
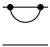
generating object: 

generating morphisms:  and 

relations:  and  $= -2$ 

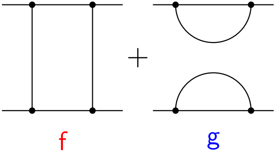
Proposition:

There is a $\mathbb{C}$-linear monoidal functor $\mathcal{F} : \mathcal{TL} \to \mathrm{Vect}_{\mathbb{C}}$ sending

 $\longmapsto V$,  $\longmapsto n$,  $\longmapsto u$

Example 1

Draw the map from $V \otimes V$ to $V \otimes V$ sending $x \otimes y$ to $y \otimes x$ as element of $\mathcal{TL}$.



f: $\mathrm{id}_V \otimes \mathrm{id}_V \circ \mathrm{id}_V \otimes \mathrm{id}_V : V \otimes V \to V \otimes V$

g: $u \circ n : V \otimes V \to V \otimes V$

Write $x = x_1 e_1 + x_2 e_2$ and $y = y_1 e_1 + y_2 e_2$

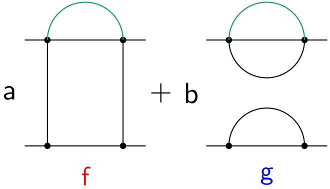
$f(x \otimes y) = x \otimes y =$
$x_1 y_1 (e_1 \otimes e_1) + x_1 y_2 (e_1 \otimes e_2) + x_2 y_1 (e_2 \otimes e_1) + x_2 y_2 (e_2 \otimes e_2)$

$g(x \otimes y) = x_2 y_1 (e_1 \otimes e_2) - x_2 y_1 (e_2 \otimes e_1) - x_1 y_2 (e_1 \otimes e_2) + x_1 y_2 (e_2 \otimes e_1)$

$(f + g)(x \otimes y) = f(x \otimes y) + g(x \otimes y) =$
$y_1 x_1 (e_1 \otimes e_1) + y_1 x_2 (e_1 \otimes e_2) + y_2 x_1 (e_2 \otimes e_1) + y_2 x_2 (e_2 \otimes e_2) = y \otimes x$

Example 2

Find an endomorphism of two strands which is killed by a cap on top.



a  $-2b$  $= 0$

$(a - 2b)$  $= 0$

$\Rightarrow a = 2b$

Check:

Write $x = x_1 e_1 + x_2 e_2$ and $y = y_1 e_1 + y_2 e_2$

$(n \circ f)(x \otimes y) = x_2 y_1 - x_1 y_2$
$(n \circ g)(x \otimes y) = 2(x_1 y_2 - x_2 y_1)$

$(2b(n \circ f) + b(n \circ g))(x \otimes y) = 2bx_2 y_1 - 2bx_1 y_2 + 2bx_1 y_2 - 2bx_2 y_1 = 0$

Example 2

Claim: For the choice $a = 1$ and $b = \frac{1}{2}$ our linear combination is even an idempotent.

Check:

$$(f + \tfrac{1}{2}g)(x \otimes y) = x_1y_1 \underbrace{(e_1 \otimes e_1)}_{0} + \tfrac{1}{2}x_1y_2(\underbrace{(e_1 \otimes e_2)}_{-1} + \underbrace{(e_2 \otimes e_1)}_{1}) + \tfrac{1}{2}x_2y_1(\underbrace{(e_2 \otimes e_1)}_{1} + \underbrace{(e_1 \otimes e_2)}_{-1}) + x_2y_2 \underbrace{(e_2 \otimes e_2)}_{0}$$

$$f((f + \tfrac{1}{2}g)(x \otimes y)) = (f + \tfrac{1}{2}g)(x \otimes y)$$

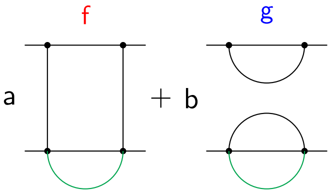
$$g((f + \tfrac{1}{2}g)(x \otimes y)) = (u \circ n)((f + \tfrac{1}{2}g)(x \otimes y)) = u(0) = 0$$

$$(f + \tfrac{1}{2}g) \circ (f + \tfrac{1}{2}g)(x \otimes y) = (f + \tfrac{1}{2}g)(x \otimes y)$$

Thus, $f + \frac{1}{2}g$ is indeed an idempotent.

Example 3

Find an endomorphism of two strands which is killed by a cup on bottom.



a  −2b  = 0

(a − 2b)  = 0

$\Rightarrow a = 2b$

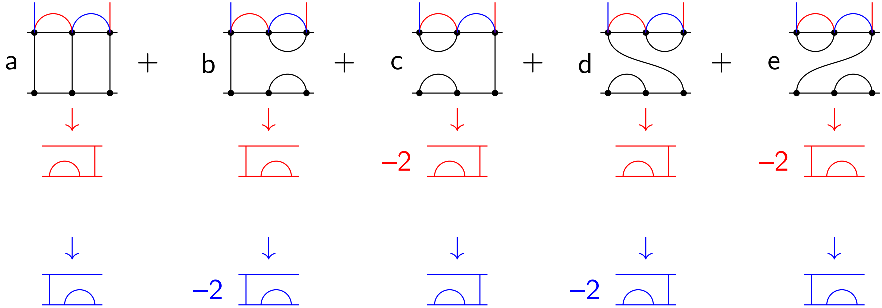
Check

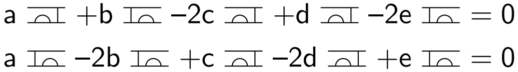
$(f \circ u)(1) = e_1 \otimes e_2 - e_2 \otimes e_1$
$(g \circ u)(1) = 2(e_2 \otimes e_1 - e_1 \otimes e_2)$

$(2b(f \circ U) + b(g \circ u))(1) =$
$2b(e_1 \otimes e_2) - 2b(e_2 \otimes e_1) + 2b(e_2 \otimes e_1) - 2b(e_1 \otimes e_2) = 0$

Example 4

Find an endomorphism of three strands which is killed by a cap on top on either of the two possible placements.





Example 4

We can write a general solution for these equations (for example in terms of e): $a = 3e, \; b = 2e, \; c = 2e, \; d = e$

Check:

$3e$  $+2e$  $-4e$  $+e$  $-2e$  $= 0$

$3e$  $-4e$  $+2e$  $-2e$  $+e$  $= 0$

This endomorphism is even an idempotent for the choice $a = 1, \; b = \frac{2}{3}, \; c = \frac{2}{3}, \; d = \frac{1}{3}, \; e = \frac{1}{3}$.

Another monoidal category

Another monoidal category

Consider $\mathrm{Vect}_{\mathbb{R}}$ with usual tensor product.

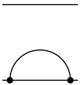
- $A := \mathbb{R}[x]/(x^2)$ We draw:

  $A =$ 

- Function $\cap$:

  $$\cap : A \otimes A \to \mathbb{R}$$
  $$f \otimes g \mapsto \text{coefficient of } x \text{ in } fg$$

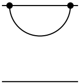
  We draw:

  $\cap :=$ 

- Function $\cup$:

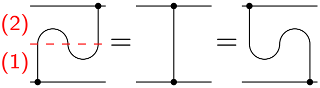
  $$\cup : \mathbb{R} \to A \otimes A$$
  $$1 \mapsto x \otimes 1 + 1 \otimes x$$

  We draw:

  $\cup :=$ 

Another monoidal category

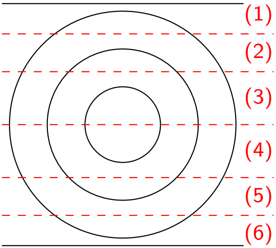
We have the following equalities:



(1): $\mathrm{id}_A \otimes \cup : A \otimes \mathbb{R} \to A \otimes A \otimes A$
(2): $\cap \otimes \mathrm{id}_A : A \otimes A \otimes A \to \mathbb{R} \otimes A$

$$\begin{aligned}
&(\cap \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \cup)(ax + b \otimes 1) \\
&= (\cap \otimes \mathrm{id}_A)(ax + b \otimes (x \otimes 1 + 1 \otimes x)) \\
&= (\cap \otimes \mathrm{id}_A)(ax \otimes x \otimes 1 + ax \otimes 1 \otimes x + b \otimes x \otimes 1 + b \otimes 1 \otimes x) \\
&= (\cap \otimes \mathrm{id}_A)(a(x \otimes x \otimes 1) + a(x \otimes 1 \otimes x) + b(1 \otimes x \otimes 1) + b(1 \otimes 1 \otimes x)) \\
&= a(1 \otimes x) + b(1 \otimes 1) \\
&= 1 \otimes ax + 1 \otimes b \\
&= 1 \otimes ax + b
\end{aligned}$$

Another monoidal category



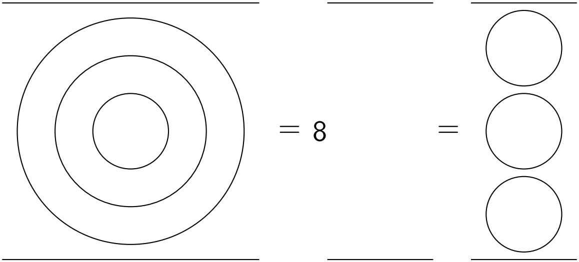
(1): $\cap$
(2): $id_A \otimes \cap \otimes id_A$
(3): $id_A \otimes id_A \otimes \cap \otimes id_A \otimes id_A$
(4): $id_A \otimes id_A \otimes \cup \otimes id_A \otimes id_A$
(5): $id_A \otimes \cup \otimes id_A$
(6): $\cup$

$$
\begin{aligned}
1 &\xrightarrow{(6)} x \otimes 1 + 1 \otimes x = x \otimes 1 \otimes 1 + 1 \otimes 1 \otimes x \\
&\xrightarrow{(5)} x \otimes (x \otimes 1 + 1 \otimes x) \otimes 1 + 1 \otimes (x \otimes 1 + 1 \otimes x) \otimes x \\
&\xrightarrow{(4)} x \otimes x \otimes (x \otimes 1 + 1 \otimes x) \otimes 1 \otimes 1 + x \otimes 1 \otimes (x \otimes 1 + 1 \otimes x) \otimes x \otimes 1 \\
&\quad + 1 \otimes x \otimes (x \otimes 1 + 1 \otimes x) \otimes 1 \otimes x + 1 \otimes 1 \otimes (x \otimes 1 + 1 \otimes x) \otimes x \otimes x \\
&\xrightarrow{(3)} 2(x \otimes x \otimes 1 \otimes 1) + 2(x \otimes 1 \otimes x \otimes 1) \\
&\quad + 2(1 \otimes x \otimes 1 \otimes x) + 2(1 \otimes 1 \otimes x \otimes x) \\
&\xrightarrow{(2)} 4(x \otimes 1) + 4(1 \otimes x) \\
&\xrightarrow{(1)} 8
\end{aligned}
$$

Another monoidal category

What is $\cap \circ \cup$?

$$(\cap \circ \cup)(1) = \cap(x \otimes 1 + 1 \otimes x) = 2$$



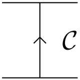
More isotopy

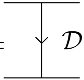
More isotopy

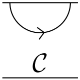
Setting:

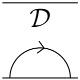
- 2-category $\mathcal{C}at$ where objects are categories
- categories $\mathcal{C}$ and $\mathcal{D}$
- functors $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ and $\mathsf{F}' : \mathcal{D} \to \mathcal{C}$ s.t. F is a left adjoint and F' is a right adjoint
- unit $\eta : \mathbb{1}_{\mathcal{C}} \to \mathsf{F}'\mathsf{F}$ and counit $\varepsilon : \mathsf{F}\mathsf{F}' \to \mathbb{1}_{\mathcal{D}}$

We draw:

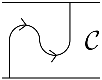
$\mathsf{F} :=$ 

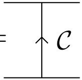
$\mathsf{F}' :=$ 

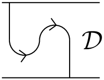
$\eta :=$ 

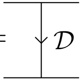
$\varepsilon :=$ 

Axioms of adjunction:



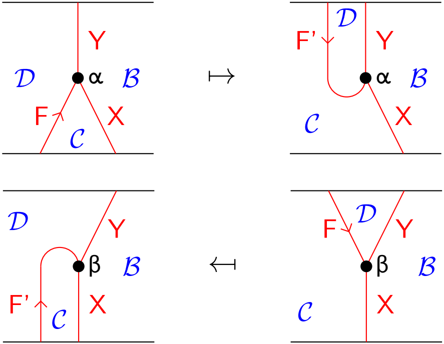
$=$


and


$=$


More isotopy

Let $\mathcal{B}$ be another category and $X : \mathcal{B} \to \mathcal{C}$ and $Y : \mathcal{B} \to \mathcal{D}$ functors. Then we can view the adjunction $F \dashv F'$ as bijection of 2-morphism spaces $\mathrm{Hom}(FX, Y) \cong \mathrm{Hom}(X, F'Y)$ given by:



If $\mathcal{B}$ is a category with only one object and the identity morphism, X and Y can be seen as objects in $\mathcal{C}$ resp. $\mathcal{D}$. Then we are left with the familiar bijection of Hom-spaces of adjunctions.

More isotopy

Assume from know on that F and F' are biadjoints (i.e. F is also a right adjoint and F' also a left adjoint).

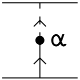
Natural transformations:

Relations:

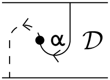
More isotopy

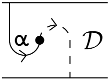
Let $G : \mathcal{C} \to \mathcal{D}$ be another biadjoint functor i.e. 1-morphism with adjoint G'. Draw G, G', unit and counit similar to F, just with dashed lines.

Definition

Let $\alpha : F \to G$ be a 2-morphism, thus we draw: 

We get two 2-morphisms $'\alpha, \alpha' : G' \to F'$ called **left** resp. **right mates of** α with respect to the chosen biadjunction given by:

$'\alpha :=$ 

$\alpha' :=$ 

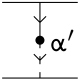
Remark:

In general: $'\alpha \neq \alpha'$!

More isotopy

Definition:

We say α is **cyclic**, if $'\alpha = \alpha'$.

Then we can draw unambiguously: 

Proposition:

If all 1-morphisms have biadjoints and all 2-morphisms are cyclic, then (using the above convention of drawing) diagrams represent a 2-morphism up to true isotopy unambiguously.