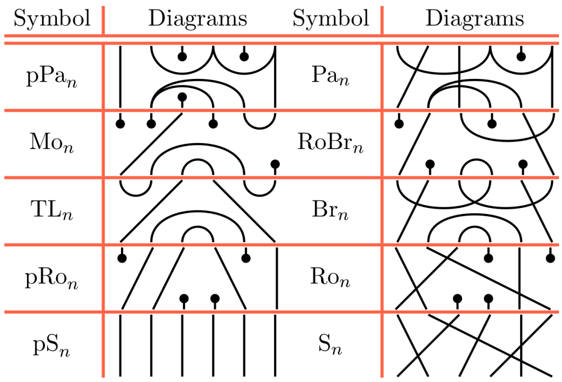
Exercises 2

1. Diagram algebras and H-reduction

Recall the arguably most important diagram monoids:

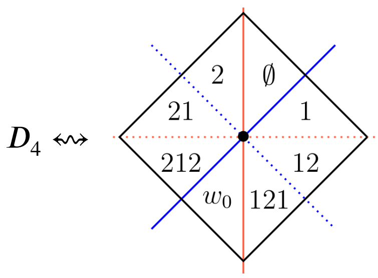


Fix some field $\mathbb{K}$. In all cases, the respective algebras are obtained by evaluating floating components to a fixed $\delta \in \mathbb{K}$. (If that doesn't make sense to you, then I have messed up: my bad...)

a) Classify the simple modules for your favorite(s) of these diagram algebras.
b) (') If you know the quantum versions of these algebras, such as the BMW algebra, then try those as well.

2. Finite fun with dihedral groups – classical

Let $\emptyset$ denote the unit and let $D_n = \langle 1, 2 | 1^2 = 2^2 = (12)^n = \emptyset \rangle$ be the dihedral group of the n gon.



a) Use *e.g.* the Magma online calculator (see below) to guess the classification of simple D_n-modules over $\mathbb{C}$. You can use the code

```
n:=5;
CharacterTable(DihedralGroup(n))
```

$n = 4:$

```
----------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  2  2  4
----------------------
p  =  2   1  1  1  1  2
----------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1  1 -1
X.3   +   1  1  1 -1 -1
X.4   +   1  1 -1 -1  1
X.5   +   2 -2  0  0  0
```

, $n = 5:$

```
-----------------------
Class |   1  2    3    4
Size  |   1  5    2    2
Order |   1  2    5    5
-----------------------
p  =  2   1  1    4    3
p  =  5   1  2    1    1
-----------------------
X.1   +   1  1    1    1
X.2   +   1 -1    1    1
X.3   +   2  0   Z1 Z1#2
X.4   +   2  0 Z1#2   Z1
```

and vary n.

b) Show that your guessed classification is true.

c) (*) What happens for general fields?

3. Infinite fun with dihedral groups – à la KL

Retain the notation from Exercise 2. For a field $\mathbb{K}$ consider the group algebra $S = \mathbb{K}[D_\infty]$ of the infinite dihedral group $D_\infty = \langle 1,2 | 1^2 = 2^2 = \emptyset \rangle$. Every element of D_∞ has a unique reduced expression. We write $k,1,2$ and $k,2,1$ for the reduced expressions $\dots 12$ and $\dots 21$ in k symbols.

The algebra S has a KL basis $\{b_w | w \in D_\infty\}$ (whose precise definition does not matter) with identity $b_\emptyset$. Set $b_{0,a,b} = 0$. The nonidentity multiplication rules are given by the Clebsch–Gordan formula:

$$b_{k,1,2} b_{j,1,2} = \begin{cases} 2b_{|k-j|+1,1,2} + \dots + 2b_{|k+j|-1,1,2} & \text{j,1,2=2}\dots 12, \\ b_{|k-j|,1,2} + 2b_{|k-j|+2,1,2} + \dots + 2b_{|k+j|-2,1,2} + b_{|k+j|,1,2} & \text{j,1,2=1}\dots 12. \end{cases}$$

There are also similar formulas with $b_{j,2,1}$ on the right or $b_{k,1,2}$ on the left.

For example:

$$b_{1212} b_{21212} = 2b_{12} + 2b_{1212} + 2b_{121212} + 2b_{12121212},$$
$$b_{1212} b_{121212} = b_{12} + 2b_{1212} + 2b_{121212} + 2b_{12121212} + b_{1212121212}.$$

a) Compute the cell structure of S with respect to the KL basis $\{b_w | w \in D_\infty\}$ for $\mathrm{char}(\mathbb{K}) \neq 2$. Skip the identification of the nontrivial $S_\mathcal{H}$ for now.
b) (') Compare the nontrivial $S_\mathcal{H}$ of S to the Grothendieck algebra of complex finite dimensional $\mathrm{SO}_3(\mathbb{C})$-representations.
c) What happens in characteristic two?

4. Finite fun with dihedral groups – à la KL

Retain the notation from Exercise 3. Let $S = D_n = \langle 1,2 | 1^2 = 2^2 = (12)^n = \emptyset \rangle$ be the dihedral group of the n gon. The longest element is $w_0 = n,1,2 = n,2,1$.

With respect to the KL basis and its multiplication rules, the only change compare to D_∞ is that expressions of the form (here $d > 0$)

$$b_{n-d,1,2} + b_{n+d,1,2} \longmapsto 2b_{w_0}, \quad b_{n-d,2,1} + b_{n+d,2,1} \longmapsto 2b_{w_0}.$$

are replaced as indicated. This is the truncated Clebsch–Gordan formula.

For example, for $n = 6$ one gets:

$$b_{1212} b_{21212} = 2b_{12} + \boxed{2b_{1212}} + \underline{2b_{121212}} + \boxed{2b_{12121212}} = 2b_{12} + 6b_{121212},$$
$$b_{1212} b_{121212} = \boxed{b_{12}} + \boxed{2b_{1212}} + \underline{2b_{121212}} + \boxed{2b_{12121212}} + \boxed{b_{1212121212}} = 8b_{121212}.$$

a) Compute the cell structure of S with respect to the KL basis $\{b_w | w \in D_n\}$ for $\mathbb{K} = \mathbb{C}$ and odd n. Skip the identification of the nontrivial $S_\mathcal{H}$ for now.
b) (') In Exercise 3 we have seen that the representation theory of the infinite dihedral group for the middle cell is controlled by $\mathrm{SO}_3(\mathbb{C})$. Show that the same is true in finite type when working with an appropriate semisimplification of $\mathrm{SO}_3(\mathbb{C})$-representations.
c) (*) What are the nontrivial $S_\mathcal{H}$ explicitly?
d) What is the difference between odd and even n?
e) (*) What happens over general fields?

- There might be typos on the exercise sheets, my bad. Be prepared.
- Star exercises are a bit trickier; prime exercises use notions I haven't explained.

Exercises - hints and remarks 2

SageMath online calculator `https://sagecell.sagemath.org/` with the relevant material summarized on
`https://doc.sagemath.org/html/en/thematic_tutorials/lie/weyl_character_ring.html`
Magma online calculator `http://magma.maths.usyd.edu.au/calc/`

Hints for Exercise 2

The one dimensional representations are easy to construct. For the two dimensional representations use

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 2 \longmapsto \begin{pmatrix} \cos(2\pi k/n) & -\sin(2\pi k/n) \\ -\sin(2\pi k/n) & -\cos(2\pi k/n) \end{pmatrix}.$$

Via easy calculations (seriously: these are 2x2 matrices!) one verifies: The matrices satisfy the relations of D_n and have no common eigenvector, so the associated representations are simple. They are also nonconjugate for $k \in \{1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$. Finally, the sum of the squares of their dimensions is $2n$, so we are done.

In general, $D_n \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, and one can use 12 and 1 as the generators of the two groups in this semidirect product. Now induce from those two groups and hope for the best.

Hints for Exercise 3

Unless the characteristic of $\mathbb{K}$ is two, the picture should look like

$$\begin{array}{c|c|c|c} \mathcal{J}_m & \begin{array}{c|c} b_1, b_{121}, \dots & b_{12}, b_{1212}, \dots \\ \hline b_{21}, b_{2121}, \dots & b_2, b_{212}, \dots \end{array} & S_{\mathcal{H}} \cong_s \mathbb{K}[\mathbb{Z}] \\ \mathcal{J}_\emptyset & b_\emptyset & S_{\mathcal{H}} \cong \mathbb{K} \end{array}$$

The Grothendieck algebra (abelian or additive, that does not make a difference) of $\mathrm{SO}_3(\mathbb{C})$ can be computed via the SageMath online calculator, see above, with the code

```
A=WeylCharacterRing(A1,style=coroots);
k=5;
j=4;
A(2*k,0)*A(2*j,0)
```

You need to vary k and j, and identify b_{121} with $A(2) = A(2,0)$ up to scaling. Neither b_{121} nor $A(2)$ satisfy any polynomial relation, but both generate the respective algebras.

Hints for Exercise 4

Unless the characteristic of $\mathbb{K}$ is nonzero and small, the picture for n being odd should look like

$$n \text{ odd}: \begin{array}{ccc} \mathcal{J}_{w_0} & b_{w_0} & S_{\mathcal{H}} \cong \mathbb{K} \\ \mathcal{J}_m & \begin{array}{c|c} b_1, b_{121}, \dots & b_{12}, b_{1212}, \dots \\ \hline b_{21}, b_{2121}, \dots & b_2, b_{212}, \dots \end{array} & S_{\mathcal{H}} \cong_s \mathbb{K}[\mathbb{Z}/\tfrac{n-1}{2}\mathbb{Z}] \\ \mathcal{J}_\emptyset & b_\emptyset & S_{\mathcal{H}} \cong \mathbb{K} \end{array}$$

That the diagonal $S_{\mathcal{H}}$ have pseudo idempotents is clear by $b_1 b_1 = 2b_1$. For the off-diagonal elements let us take $n = 7$ and $b = b_{12} - b_{1212} + b_{121212}$. Then the multiplication table

	b_{12}	$-b_{1212}$	b_{121212}
b_{12}	$2b_{12} + b_{1212}$	$-b_{12} - 2b_{1212} - b_{121212}$	$b_{1212} + b_{121212}$
$-b_{1212}$	$-b_{12} - 2b_{1212} - b_{121212}$	$2b_{12} + 2b_{1212} + b_{121212}$	$-b_{12} - b_{1212}$
b_{121212}	$b_{1212} + b_{121212}$	$-b_{12} - b_{1212}$	b_{12}

verifies that $b^2 = b$. The general case is similar. (Note that b would be an infinite alternating sum for $n = \infty$, and that is why the off-diagonal $S_{\mathcal{H}}$ do not have pseudo idempotents in the infinite case.)

The isomorphism $S_{\mathcal{H}} \cong_s \mathbb{K}[\mathbb{Z}/\frac{n-1}{2}\mathbb{Z}]$ for nonsilly $\mathbb{K}$ can be verified as follows. Let $U_k^3(X)$ be the (Chebyshev-like multiplication by quantum three) polynomial defined via $U_0^3(X) = 1$, $U_1^3(X) = X$ and

$$U_k^3(X) = (X-1)U_{k-1}^3(X) - U_{k-2}^3(X).$$

This polynomial is the defining polynomial for $SO_3(\mathbb{C})$ in the sense that $U_k^3(X)$ corresponds to the highest weight summand in the tensor product $(X = \mathbb{C}^3)^{\otimes k}$. Here is some SageMath code:

```
A=WeylCharacterRing(A1,style=coroots);
gen=A(2,0);
k=7;
def U(n,x):
if n == 0:
return 1
elif n == 1:
return x
else:
return (x-1) * U(n-1,x) - U(n-2,x)
print(U(k,gen))
```

Now $U_m^3(b_{121}) = 0$ for $m = \frac{n-1}{2}$, so $S_{\mathcal{H}} \cong_s \mathbb{K}[X]/(U_m^3(X))$. Since $U_m^3(X)$ has distinct roots, we can then rescale $\mathbb{K}[X]/(U_m^3(X))$ to $\mathbb{K}[X]/(X^m - 1) \cong \mathbb{K}[\mathbb{Z}/m\mathbb{Z}]$.

That was the case of $SO_3(\mathbb{C})$, so you need to argue why this implies the same for the KL basis of the finite dihedral group.