Exercises 1

1. Two trivial representations!?

Let S be a finite monoid.

a) Show that S has a unique bottom and top J-cell.
b) Show that both of these J-cells are idempotent.
c) Let $\mathbb{K}$ be some field and $G \subset S$ be the subgroup of all invertible elements. Then we define trivial representations (yes, a monoid has two trivial representations):

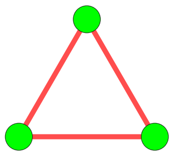
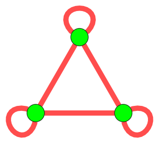
$$M_b\colon S \to \mathbb{K}, s \longmapsto \begin{cases} 1 & \text{if } s \in G, \\ 0 & \text{else,} \end{cases} \qquad M_t\colon S \to \mathbb{K}, s \longmapsto 1.$$

Identify the apexes of the simple S-modules M_b and M_t.

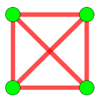
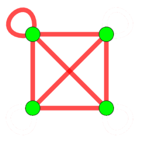
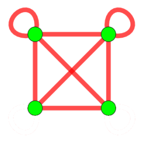
2. Endomorphisms of graphs

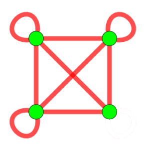
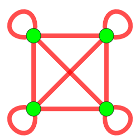
Let Γ be a graph. The set $S = \mathrm{End}(\Gamma)$ of graph endomorphisms is a monoid via composition.

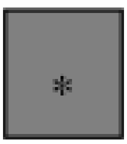
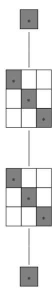
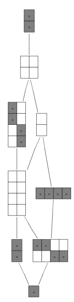
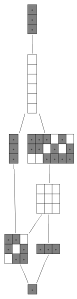
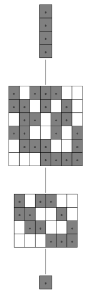
a) Compute the cell structure and classify simple modules for $S = \mathrm{End}(\Gamma)$ and $T = \mathrm{End}(\Gamma')$ for the following two graphs.

$\Gamma =$  , $\Gamma' =$ 

b) If you have done (a), then you should have seen two familiar monoids. Can you guess the general picture how they arise as graph monoids?
c) (*) Here are a few more graph monoids $S_i = \mathrm{End}(\Gamma_i)$ and their cell pictures. I do not know the general pattern; maybe someone has an idea.

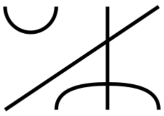
$\Gamma_0 =$  , $\Gamma_1 =$  , $\Gamma_2 =$  ,

$\Gamma_3 =$  , $\Gamma_4 =$ 

For $S_0\colon |S_0| = 24$,  , For $S_1\colon |S_1| = 34$,  , For $S_2\colon |S_2| = 56$,  , For S_3 : $|S_3| = 108$,  , For S_4 : $|S_4| = 256$, 

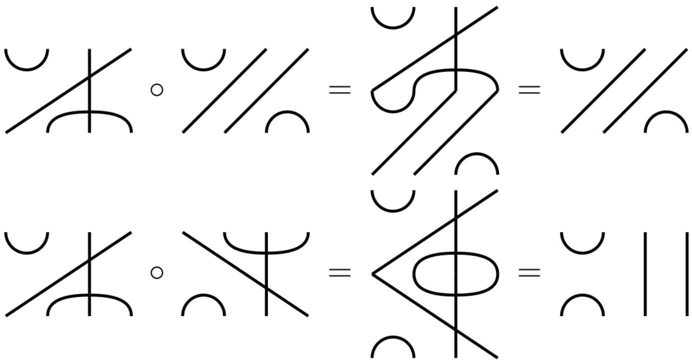
3. Diagram monoids and H-reduction

The Brauer monoid Br_n on n strands is the monoid consisting of all perfect matchings of $\{1, \dots, n\}$ with $\{-1, \dots, -n\}$, which we identify with points in the strip $\mathbb{R} \times [0, 1]$, with n points for $\{1, \dots, n\}$ at the bottom and n points for $\{-1, \dots, -n\}$ at the top line of the strip. For example

 $\leftrightsquigarrow \big\{\{1, -4\}, \{2, 4\}, \{3, -3\}, \{-1, -2\}\big\}$

Two diagrams represent the same element if and only if they represent the same perfect matching.

Stacking and removing of internal components defines a multiplication $\circ$ on Br_n, *e.g.*



(Associativity of $\circ$ is not immediate, but also not hard to see.)

a) Compute the L, R and J-cells of Br_3.
b) Compute the idempotent J-cells.
c) Compute the $\mathcal{H}(e)$-cells.
d) Parameterize the simple Br_3-modules.
e) Guess the picture for general n from the one for Br_3.

4. Binomial coefficients

Let $\mathbb{K}$ be an arbitrary field. Let $S = pRo_n$ be the planar rook monoid (the monoid of all planar partitions of $\{1, \dots, n\} \cup \{-1, \dots, -n\}$ with at most two parts and no connections within $\{1, \dots, n\}$ or $\{-1, \dots, -n\}$; see also the remarks).

a) Show that S is semisimple.
b) Compute the dimensions of the simple S-modules.

- There might be typos on the exercise sheets, my bad. Be prepared.
- Star exercises are a bit trickier; prime exercises use notions I haven't explained.

Exercises - hints and remarks 1

GAP installation guide https://www.gap-system.org/Download/index.html
Semigroups https://www.gap-system.org/Packages/semigroups.html
The code

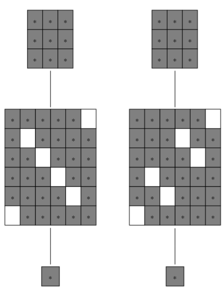
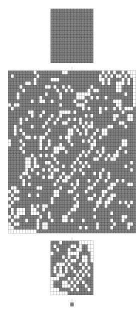
```
LoadPackage(semigroups);
S := FullTransformationMonoid(3);
FileString(t3.dot, DotString(S));
```

produces a dot string diagram of the cells of the transformation monoid on $\{1,2,3\}$. The transformation monoid can be replaced by any other monoid. (Beware: The diagrams GAP produces are flipped top-to-bottom when compared to the conventions of the lecture.)
GAP's presentation of the various monoids is often not optimal. One can vary the presentation, which gives slightly different outputs. For example,

```
LoadPackage(semigroups);
S := BrauerMonoid(4);
FileString(br4.dot, DotString(S));
```

```
LoadPackage(semigroups);
S:=Semigroup(Bipartition([[1,-1],[2,-2],[3,-3],[4,-4]]),
Bipartition([[1,-2],[2,-1],[3,-3],[4,-4]]),
Bipartition([[1,-1],[2,-3],[3,-2],[4,-4]]),
Bipartition([[1,-1],[2,-2],[3,-4],[4,-3]]),
Bipartition([[1,2],[-1,-2],[3,-3],[4,-4]]),
Bipartition([[1,-1],[2,3],[-2,-3],[4,-4]]),
Bipartition([[1,-1],[2,-2],[3,4],[-3,-4]]));
FileString(br4.dot, DotString(S));
```

both produce cells for the Brauer monoid, but the pictures are different:

(1)  , cells of Br_6 produced similarly: 

The arguably the most important diagrams monoids are:

(1) The partition monoid Pa_n is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$, the planar partition monoid pPa_n is the planar version of Pa_n.

(2) The rook Brauer monoid $RoBr_n$ is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$ with at most two parts, the Motzkin monoid Mo_n is the planar version of $RoBr_n$.

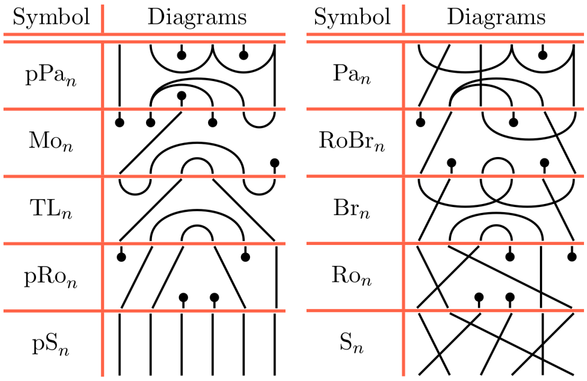
(3) The Brauer monoid Br_n is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$ with two parts, the Temperley–Lieb monoid TL_n is the planar version of Br_n.

(4) The rook monoid Ro_n is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$ with with at most two parts and no connection within $\{1,\dots,n\}$ or $\{-1,\dots,-n\}$, the planar rook monoid pRo_n is the planar version of Ro_n.

(5) The symmetric group S_n is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$ with with two parts and no connection within $\{1,\dots,n\}$ or $\{-1,\dots,-n\}$, the planar symmetric group pS_n is the planar version of S_n.

Planar = can be drawn without intersection while not leaving the defining box of endpoints.

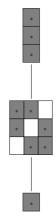
The pictures to keep in mind are the following, with the planar monoids displayed on the left:



Redo the Brauer exercise for your favorite(s) among these diagram monoids; it is fun. (The monoids pS_n and S_n are a bit boring, and my recommendation is to compute the cells for pPa_2 and TL_4.)

Hints for Exercise 2

GAP produces

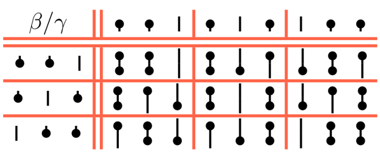
For $S: |S| = 3!$,  , For $T: |T| = 3^3$, 

Hints for Exercise 3

GAP says that Br_6 has the following statistics. We have four linearly ordered J-cells, 1+15+45+15 L and R cells. The H-cells are of sizes 6,4,2,1, so we have 10395 elements in total. The top cell consists of idempotent H-cells only, and otherwise there is at least one idempotent per J-cell, see Equation 1.

Hints for Exercise 4

For each J-cell there are bottom diagrams $\beta_1,\dots,\beta_L$ and top diagrams $\gamma_1,\dots,\gamma_R$ indexing the rows and columns of the J-cell in question. The Gram matrix is $P_{ij} = 1$ if $\beta_j \circ \gamma_i = 1$ and $P_{ij} = 0$ else. For example,



$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the Gram matrix of the second J-cell of pRo_3. Show that this implies that pRo_n is semisimple. The cell modules are then the simple modules, so their dimensions are easy to compute.