Tubbenhauer, 06.May.2019

Weigth modules I; Verma and dense modules

Question: What is a "free module"?

Motivation: Recall that "free things" are very useful to study "real things":

- a free group on X is the "minimal group" such that the elements of X can be considered as group elements

$\mathrm{Freegroup}(X) = \{x \mid x \in X\}$

→ elements are finite words and the only relations are those needed to give a group

- $(xy)z = x(yz)$
- $x1 = x = 1x$
- $xx^{-1} = 1 = xx^{-1}$

$\boxed{\text{And thats it, nothing more!}}$

Example: $\mathbb{Z}$ = free group on $\{1\}$

- A free commutative group has additionally

- $xy = yx$

Example: $\mathbb{Z}^n$ = free com

group on $\{1, \dots, n\}$

What makes these useful?

$$\boxed{\text{Crucial fact: Every group is a quotient of a free group!}}$$

What a free module is depends a bit on the setup.

Example: The regular rep R of a finite group G is a free module gen. by one element x.

Said otherwise:

$$R = \mathbb{K}[G] \otimes_{\mathbb{K}\dots} \{x\}$$

Induced module

where triv = trivial group acts on x trivially.

What we certainly want is what is called a weight module.

<u>Def</u>: A weight module V is a $\mathfrak{g}$-module on which $\mathfrak{h}$ acts diagonalizably.

Note that we do not require V to be fin-dim!

Example: By Jordan's theorem all fin-dim $\mathfrak{g}$-modules are weight.

However: There is no "∞-dim

"Jordan theorem". So we need to add the assumption of being weight?

More hands on:

V weight $\rightsquigarrow$

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$$

eigenvalues called weights

$$V_\lambda = \{ v \in V \mid H(v) = \lambda \cdot v \}$$

eigenspaces called weight spaces

$$\operatorname{supp} V = \{ \lambda \in \mathbb{C} \mid V_\lambda \neq 0 \}$$

support

Example: $\operatorname{supp} V^{(n)} = \{1-n, \ldots, n-1\}$

Proposition:

i) Submodules of weight modules are weight.

ii) Quotients of weight modules are weight

iii) Sums of weight modules are weight

iv) Finite tensor products of weight modules are weight

Proof: iii) Clear by $H(v \oplus w) = H(v) + H(w)$ Eigenvalue λ

iv) Clear by $H(v \otimes w) = H(v) \otimes w + v \otimes H(w)$

Eigenvalue $\lambda + \mu$

ii) Clear since any eigenbasis of V gives one for V/W

i) See the book → explicit construction of eigenvalues.

In fact:

$$(V \oplus W)_\lambda = V_\lambda \oplus W_\lambda$$

$$(V \otimes W)_\lambda = \bigoplus_{\mu+\mu'=\lambda} V_\mu \otimes V_{\mu'}$$

$$\mathrm{Supp}(V \oplus W) = \mathrm{Supp}(V) \cup \mathrm{Supp}(W)$$

$$\mathrm{Supp}(V \otimes W) = \mathrm{Supp}(V) + \mathrm{Supp}(W)$$

Question: What are the simples in the category $\mathcal{W}$ of weight modules?

"Hereby" it is helpful to have:

$$\mathcal{W} = \bigoplus_{\xi \in \mathbb{C}/2\mathbb{Z}} \mathcal{W}^{\xi}$$

$\hookrightarrow$ $\mathcal{W}^{\xi}$ has objects V with supp $V \subset \xi$

$\hookrightarrow$ $\xi \in \mathbb{C}/2\mathbb{Z}$ since

$E V_{\lambda} \subset V_{\lambda+2}$ $\quad F V_{\lambda} \subset V_{\lambda-2}$

$\Rightarrow$ It suffices to study $\mathcal{W}^{\xi}$ for a fixed $\xi \in \mathbb{C}/2\mathbb{Z}$

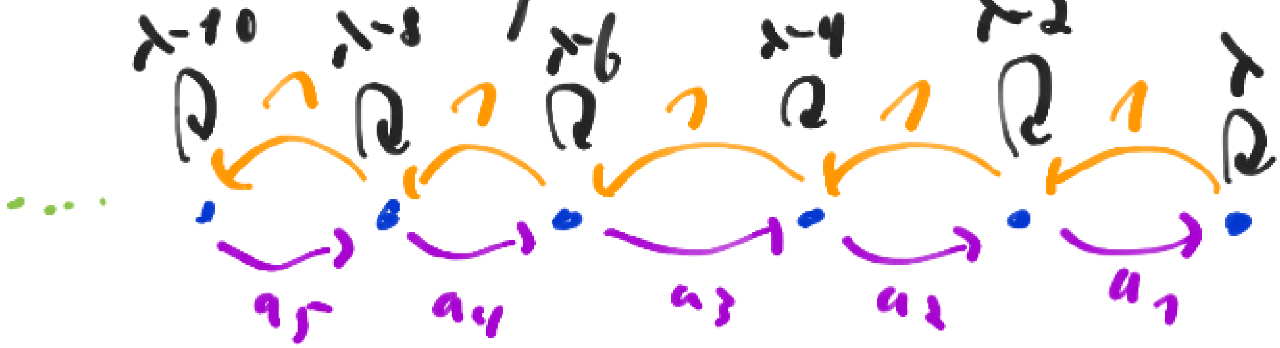
Example: $\xi = 0 \rightsquigarrow$ Even support and $V^{(\text{even})} \in \mathcal{W}^{\xi}$

$\xi = 1 \longrightarrow$ Odd support and

$V^{(odd)} \in W^5$

Verma modules = free modules for the action of F, but not E or H!

Three definitions:



The only possible choice for a_i is $a_i = i(\lambda - i + 1)$

↳ This defines $M(\lambda)^{(dual?)}$

and

supp $M(\lambda) = \{\lambda - 2i \mid i \in \mathbb{N}\}$

$-c \otimes M(\lambda)$ by $(\lambda+1)^2$

Second def:

no e's $\quad h=\lambda$ scale

$$M(\lambda) = U(\mathfrak{g})/(e,\ h-\lambda)$$

↳ Thus, elements of $M(\lambda)$ are $\{f^i \mid i \in \mathbb{N}\}$ by PBW

No other relations?

Third def: $\{\lambda\}$ ↪ h acts by λ, ↪ e acts by zero

highest weight

$$M(\lambda) = U(\mathfrak{g}) \otimes_{Borel} \{\lambda\}$$

Proposition: All definitions agree?

So $M(\lambda)$ is the $\mathfrak{g}$-module which is free for the action of f

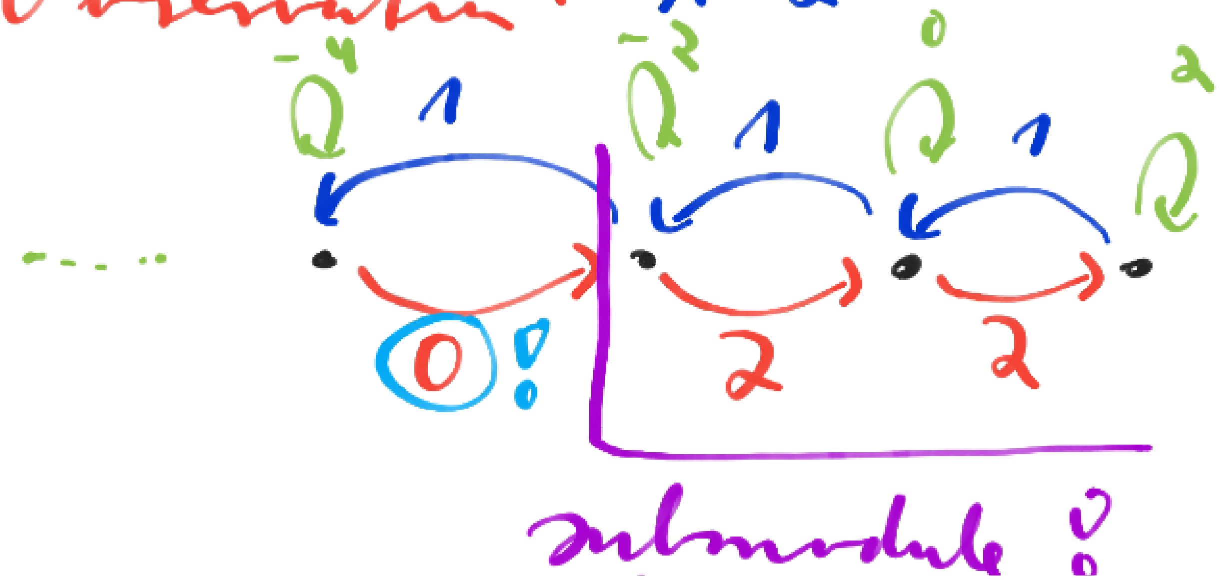
Indeed:

Proposition (Universal property)

Every $\mathfrak{g}$-module generated by λ as a highest weight module is a quotient of $M(\lambda)$

Every $\mathfrak{g}$-module with $E(v)=0$, $H(v)=\lambda$ has a unique map $M(\lambda) \longrightarrow V$

Observation: $\lambda = 2$



$V^{(2)}$

This is true in general:

$\boxed{\text{Classification of simple highest weight modules?}}$

Theorem: For every $\lambda \in \mathbb{C}$ there exists a simple $L(\lambda)$ of highest weight λ.

In fact: $\boxed{\text{All}}$ and simple are of this form.

$$L(\lambda) = \begin{cases} M(\lambda), & \lambda \notin \mathbb{N} \\ V^{(\lambda)}, & \lambda \in \mathbb{N} \end{cases}$$

Proof: That $M(\lambda)$ is simple $\iff \lambda \notin \mathbb{N}$ follows as above. Clearly $V^{(\lambda+1)}$ is

a quotient of $M(\lambda)$ for $\lambda \in \mathbb{N}$.
Finally, universal prop gives the claim.

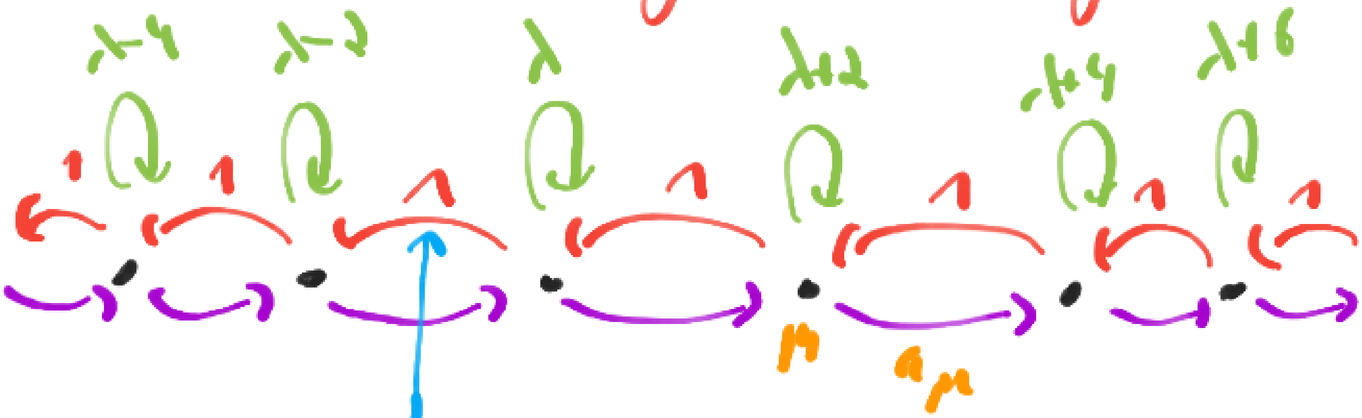
Dually: For each $\lambda \in \mathbb{C}$ there exists a simple $\bar{L}(\lambda)$ of lowest weight λ. In fact,

$$\bar{L}(\lambda) = \begin{cases} \bar{M}(\lambda) & \lambda \notin \mathbb{N} \\ V^{(\lambda+1)} & \lambda \in \mathbb{N} \end{cases}$$

dual Verma

$\boxed{\text{All}}$ simple lowest weight modules are of this form.

Now: Dense modules = "free modules with respect to the action of e and f



What can work for a_μ?

↪ our choice of normalization

Lemma: The only a_μ which work are

$\in \mathbb{C}$ $\qquad \in \xi$

$$a_\mu = \frac{1}{4}\left(T - (\mu+1)^2\right)$$

$\lambda \in \xi$

⇝ $V(\xi, T)$ denote module where c acts as T

choice of eigenvalue of c

The big theorem.

Let $g_\tau(\lambda) = \tau - (\lambda+1)^2 \in \mathbb{C}[\lambda]$

$V(\xi, \tau)$ is simple $\iff$ roots of $g_\tau \cap \xi = \emptyset$

New simples

$V(\xi, \tau)$ contains a unique simple submodule $M(\mu)$ + $V(\xi,\tau)/M(\mu) = \bar{M}(\mu+1)$

$\iff$

highest + lowest weight simples

$\mu \in$ roots of g_τ and the other root is not in ξ

$V(\xi, \tau)$ contains both roots of g_τ; and both are
in

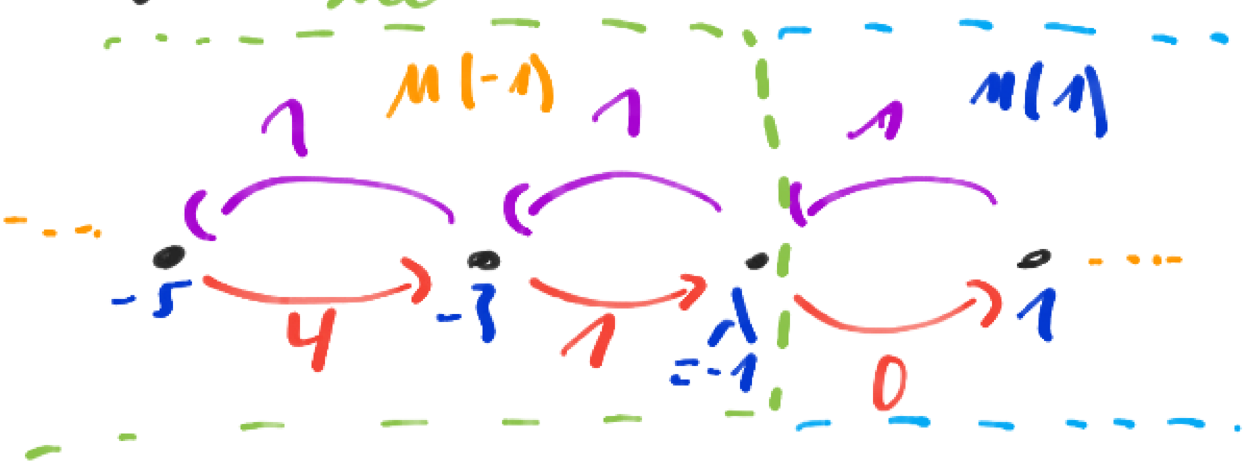
different $\mu_1 \neq \mu_2$

$\iff$

$J = n^2$ $\quad \mu_1 = n-1, \ \mu_2 = -n+1$

$\rightsquigarrow$ we get the f.d. simples

Proof: Picture $J=0$



roots of g_J are

$$\lambda_{1,2} = \pm\sqrt{J} - 1$$

$\rightsquigarrow \lambda_1 = \lambda_2 \iff J = 0$

Next time: All simples in W are of this form.

Last modified: 20:18