The dihedral cathedral

Recall: The one-color calculus

Coxeter group s labels are colors

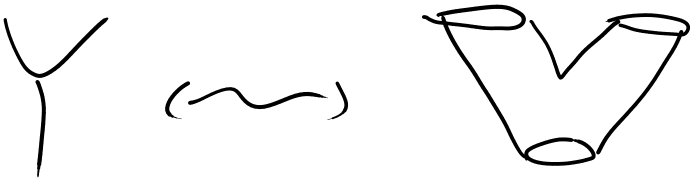
One-color: $\mathbb{C}$-linear, monoidal cat

gen by: Object $s \rightsquigarrow B_s$

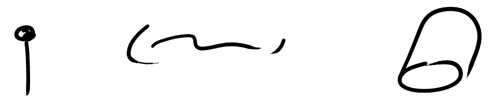
Reading ↑

Mor:

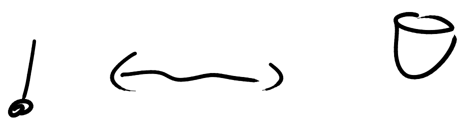
merge



split



"dots"

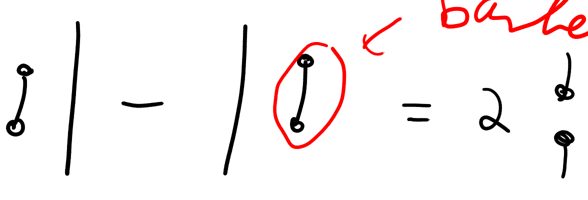


"dots"

Relations: - Frobenius relation

- nilpotency

- sliding relation



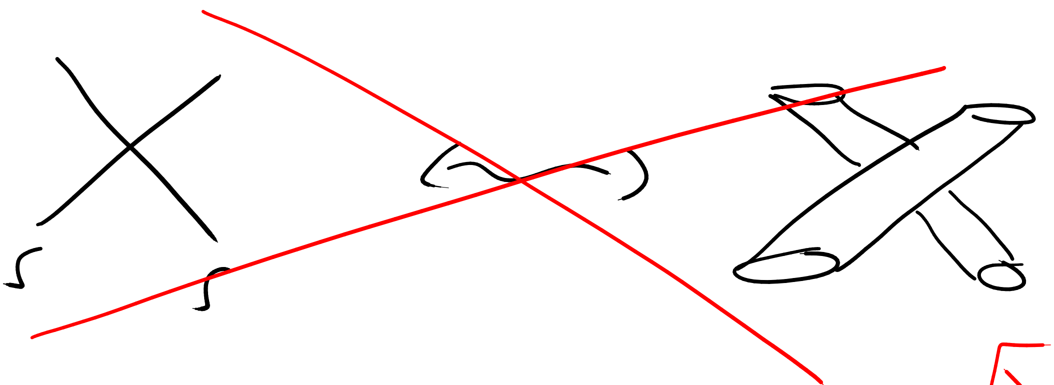
barbell

Topological relations

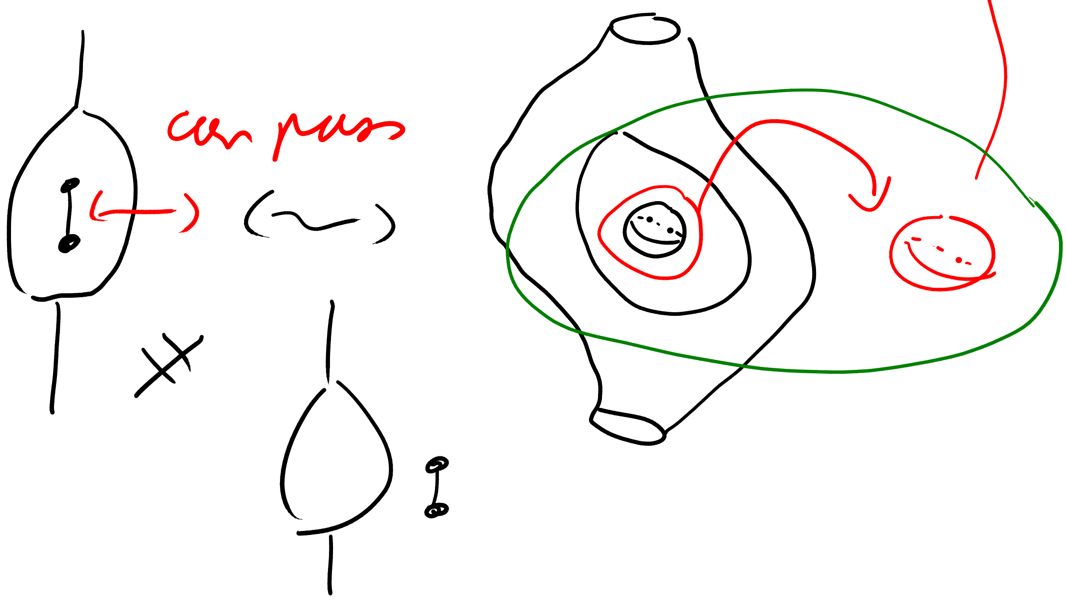
One catch

B_s, s is a Frobenius object,

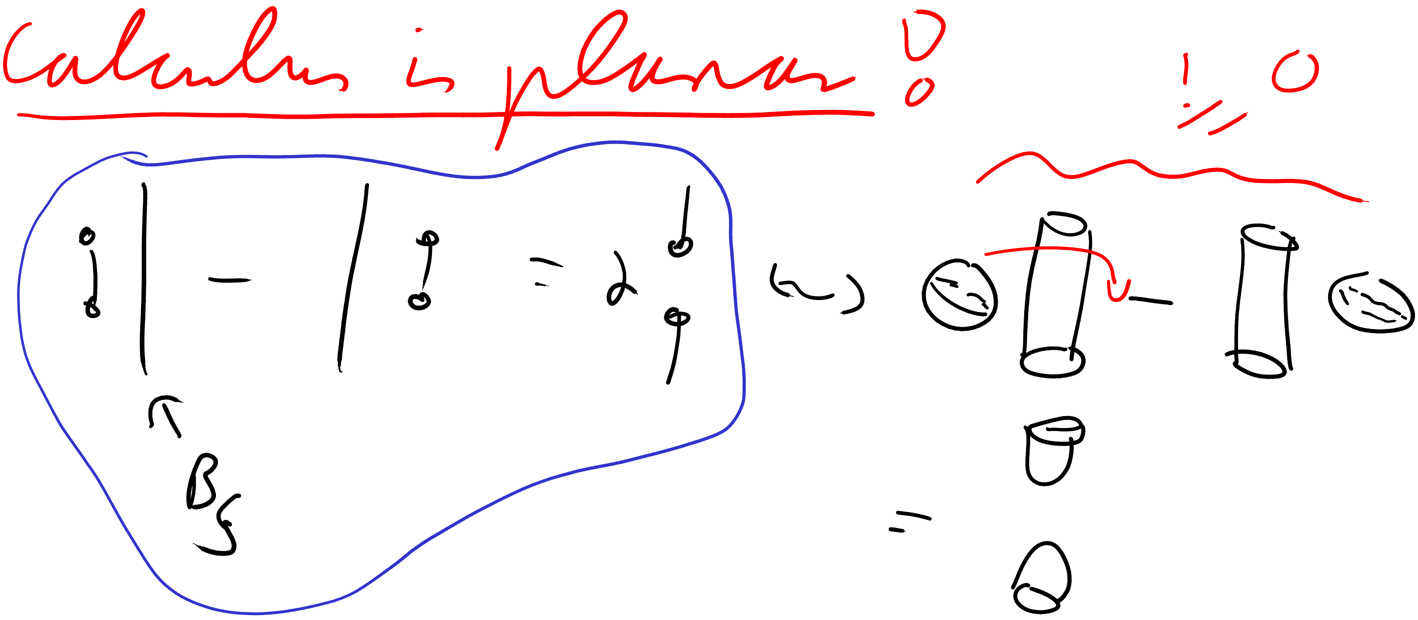
but its not symmetric, i.e.



In particular:



Calculus is planar!



Remark: Why do we want that?

$B_s \simeq R \otimes_{R^s} R$

$f \,\vdots\, g$

only s-symmetric polynomial pass freely

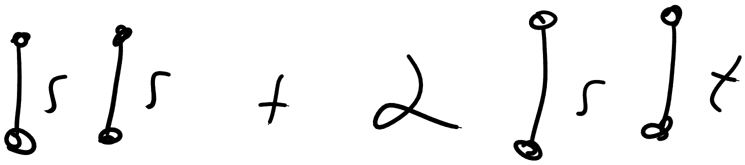
We need a planar picture!

Book's definition: allow "boxes" $\boxed{f}$ $\quad f \in R = \mathbb{C}[\alpha_s \mid s \in S]$

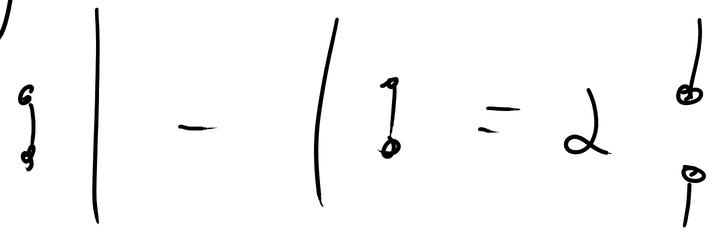
My notation:

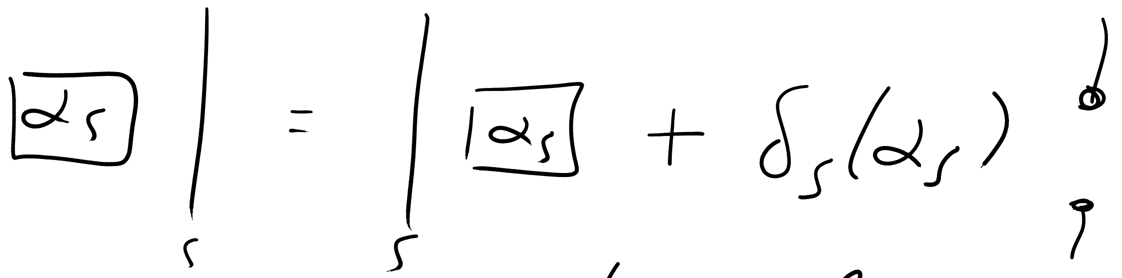
 $\rightsquigarrow \alpha_s$ monomial generators of R

$f \in R$ e.g. $f = \alpha_s^2 + 2\alpha_s\alpha_t$

$\boxed{f} =$ 

eg:

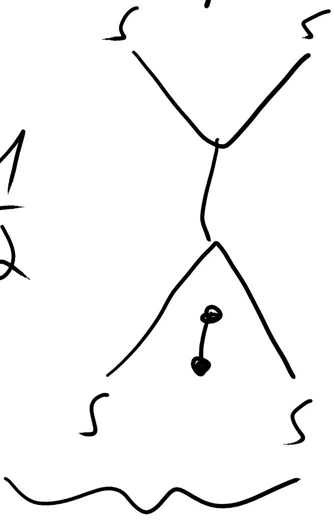




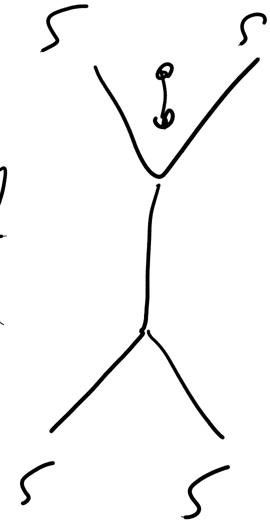
(c_s elements)

~

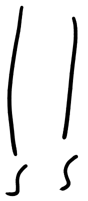
Example:

$\frac{1}{2}$ 

e_1

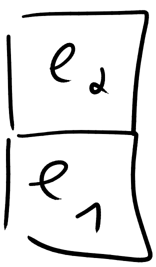
$\frac{1}{2}$ 

e_2

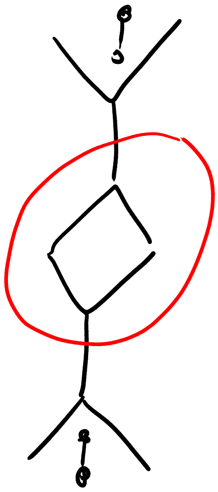


$= e_1 + e_2$

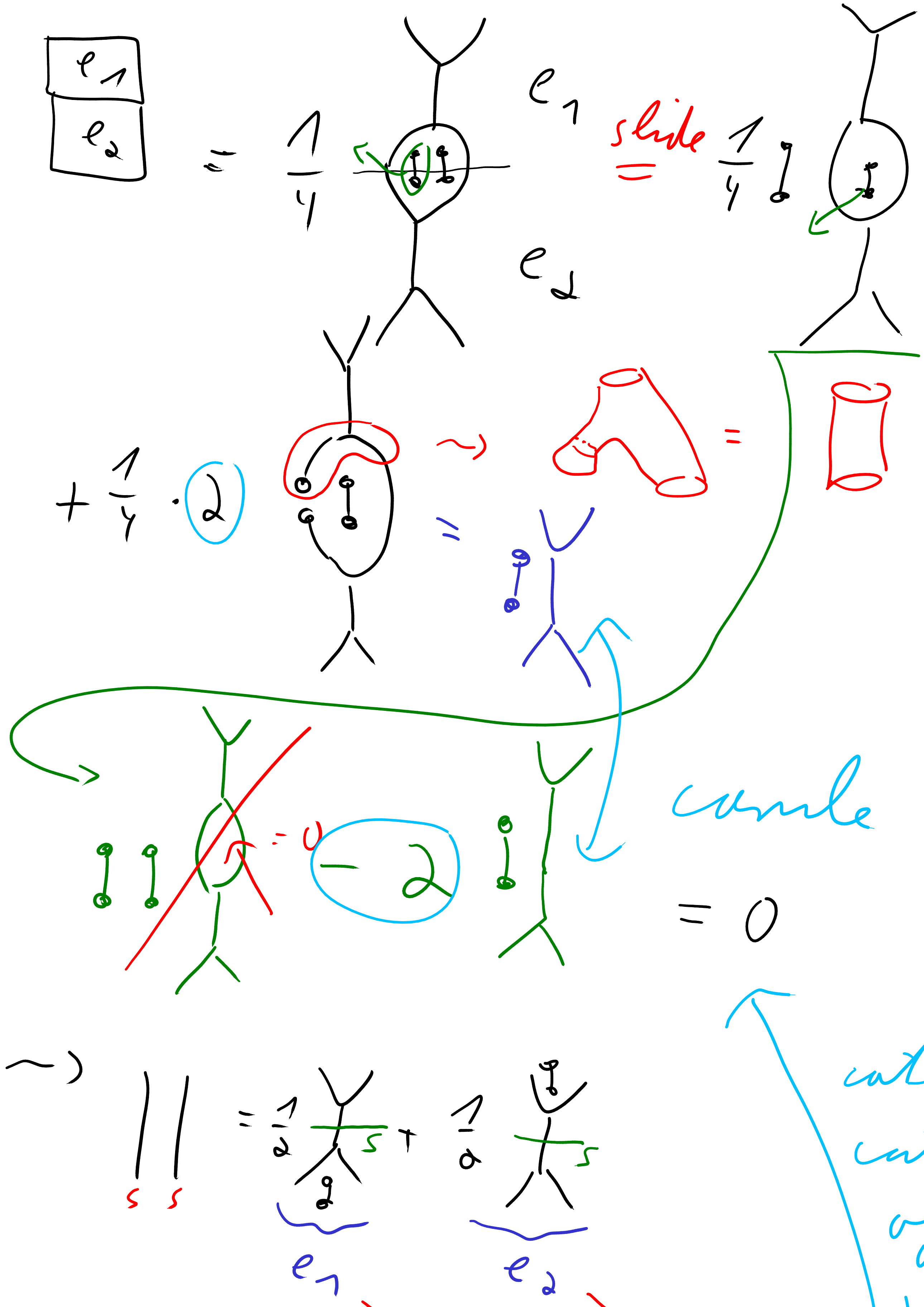
e_1, e_2 are orthogonal idempotents!



$e_2 e_1$

$\frac{1}{4}$  $= 0$

handle



$\Rightarrow B_s B_s \simeq$ Some $\oplus$ thing

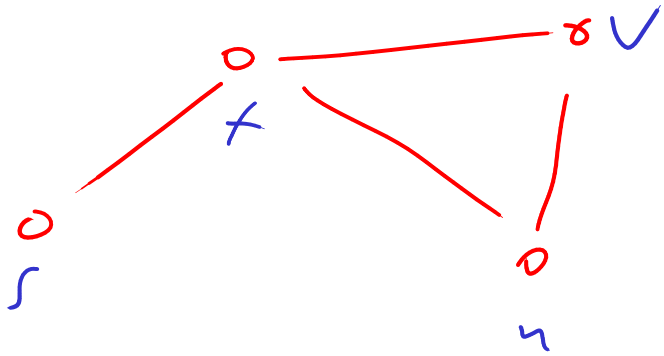
$\simeq B_s(1) \oplus B_s(-1)$

$\rightsquigarrow b_s b_s = (v+v^{-1})\, b_s$

categorification of $b_s^2 = (v+v^{-1})b_s$

Now two (or more) colors

Coxeter diagram W



Define the universal Soergel category associated to W

$\mathbb{C}$-linear, monoidal gen by

Objects colors

Morphisms: as above, but all possible monochromatic colors?

Objects in this category are finite sequences of colors

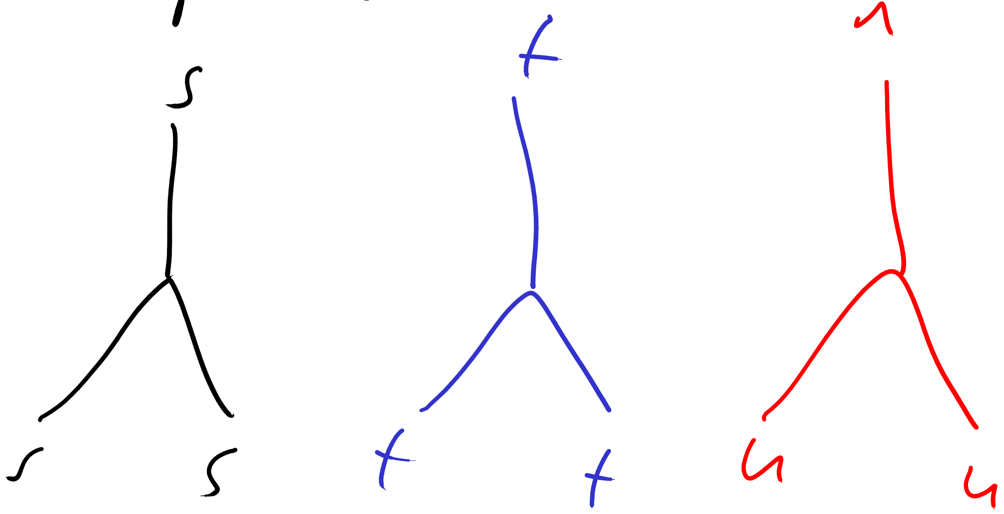
e.g.

$$s\ t\ s\ u\ s\ t\ u$$

$$\updownarrow$$

$$B_s\ B_t\ B_s\ B_u\ B_s\ B_t\ B_u$$

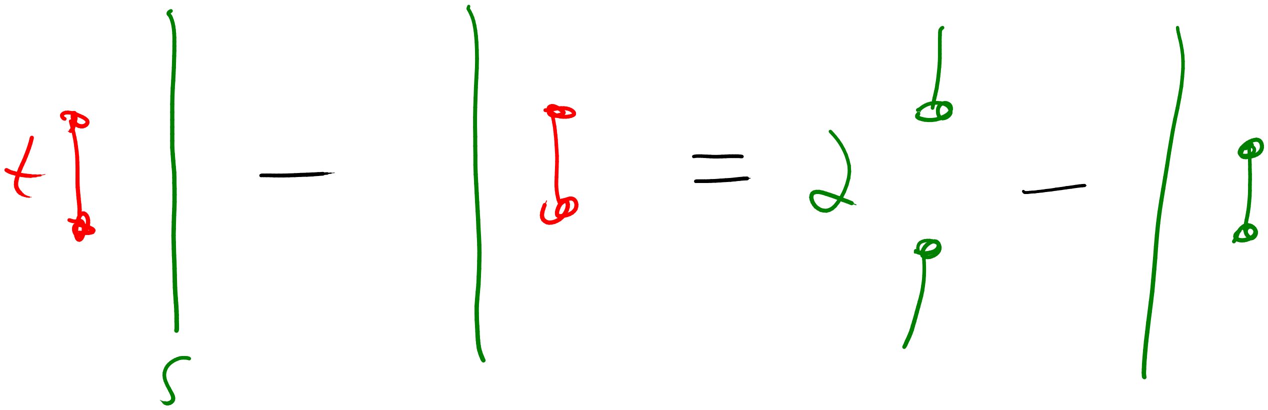
Morphisms:



etc.

Relations: as before

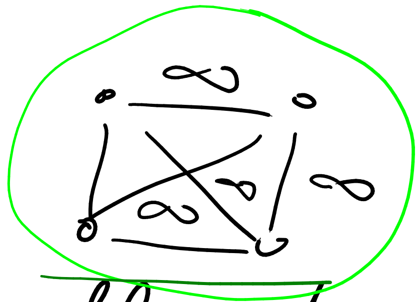
but:



Call this category

$\mathcal{D}_{BS}$ (universal)

<u>Theorem</u>: $\mathcal{D}_{BS}$ (universal)

$\simeq \mathbb{BS}(W)$ $\quad W =$ 

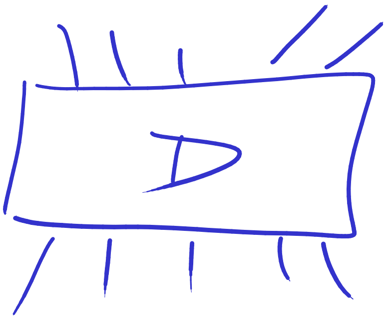
all edges labelled ∞

$\Rightarrow$ after $\oplus$, $\circledcirc$ closure

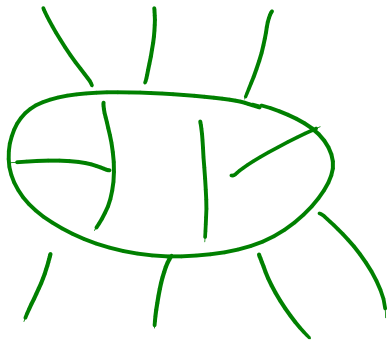
$$\mathcal{D}_{BS}(\text{universal}) \underset{\oplus,\, \circledcirc}{\simeq} \mathbb{S}Bim(W)$$

Example (how to work with D_{BS} / universal)

–



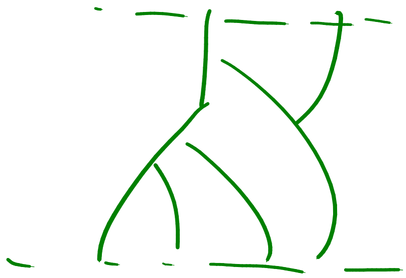
without floating components



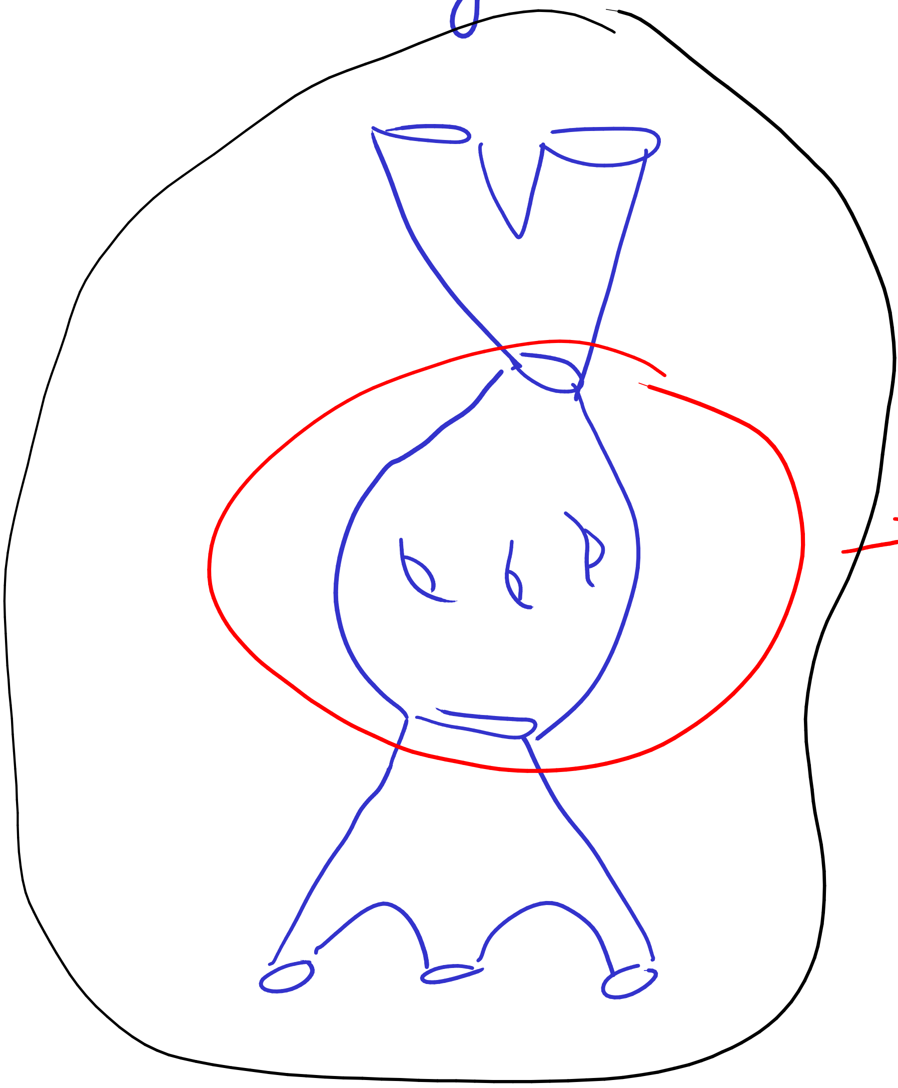
$\Rightarrow$ $D = 0$ (because handles die)

or D is a tree

$\hookrightarrow$ in the last case $D \cong D'$ tree as long as the boundary match



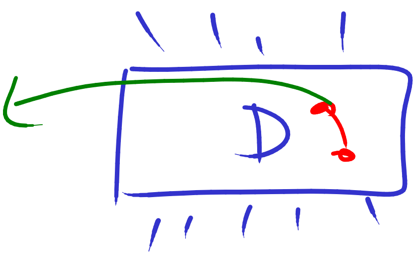
Normal form



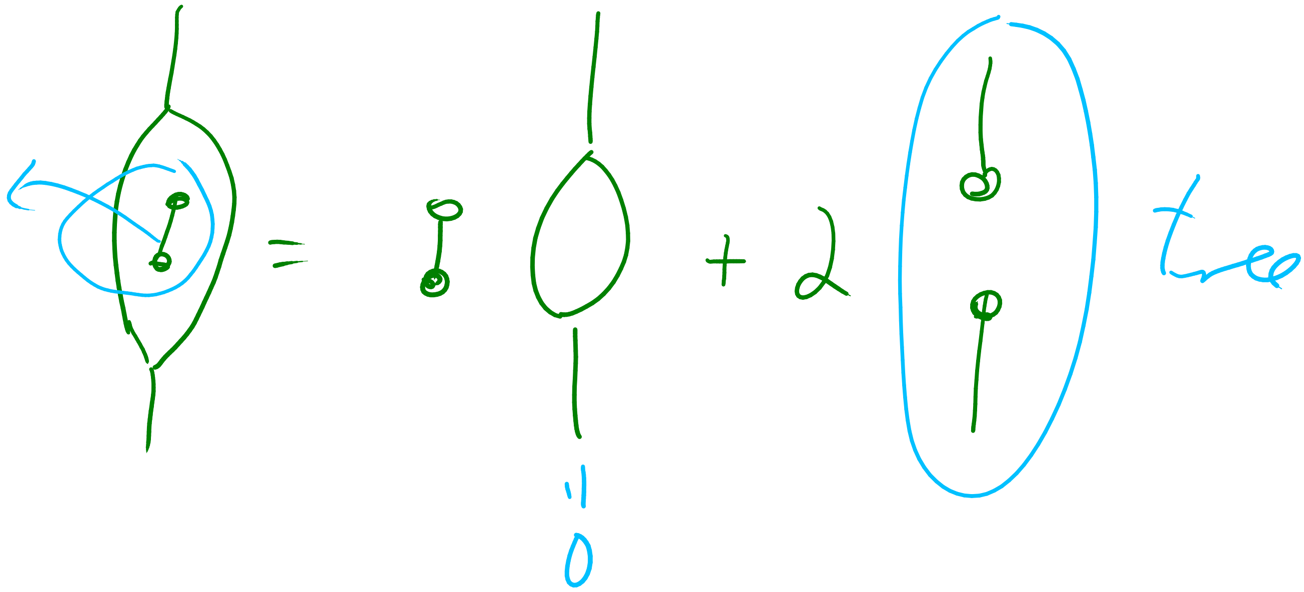
$\rightarrow g = 0$ if $D \neq 0$

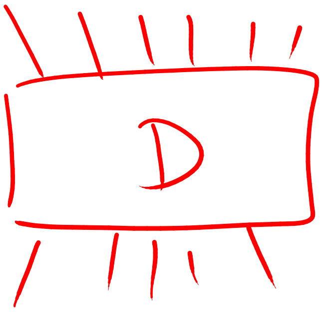
$\rightsquigarrow$ number of merges and splits determines boundary

– any polynomial in



can be slid out up to "easier diagrams"

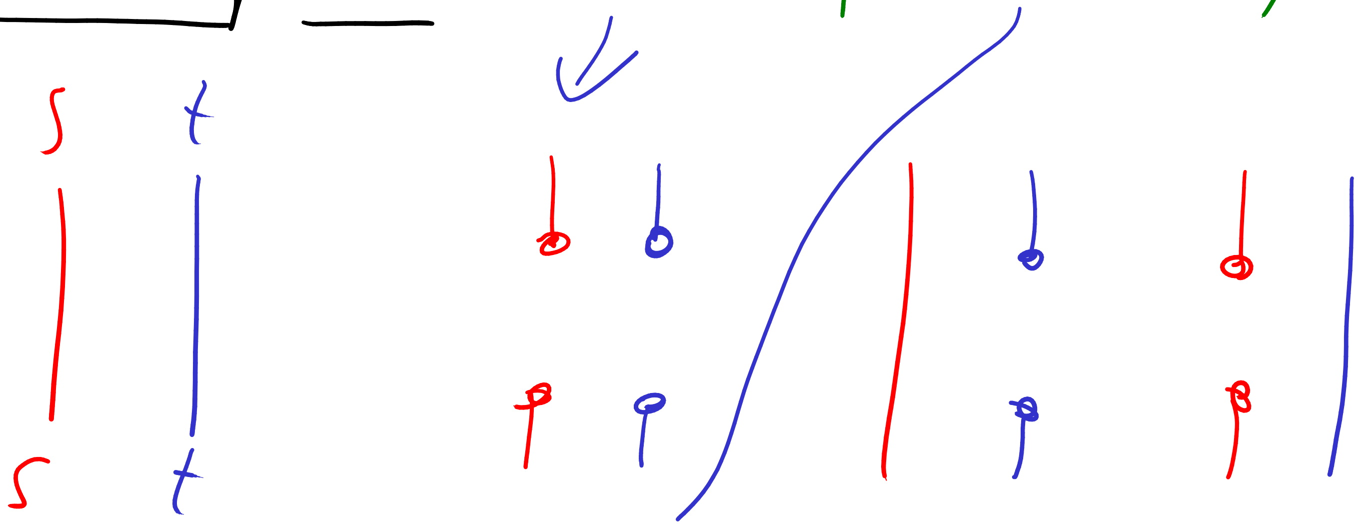




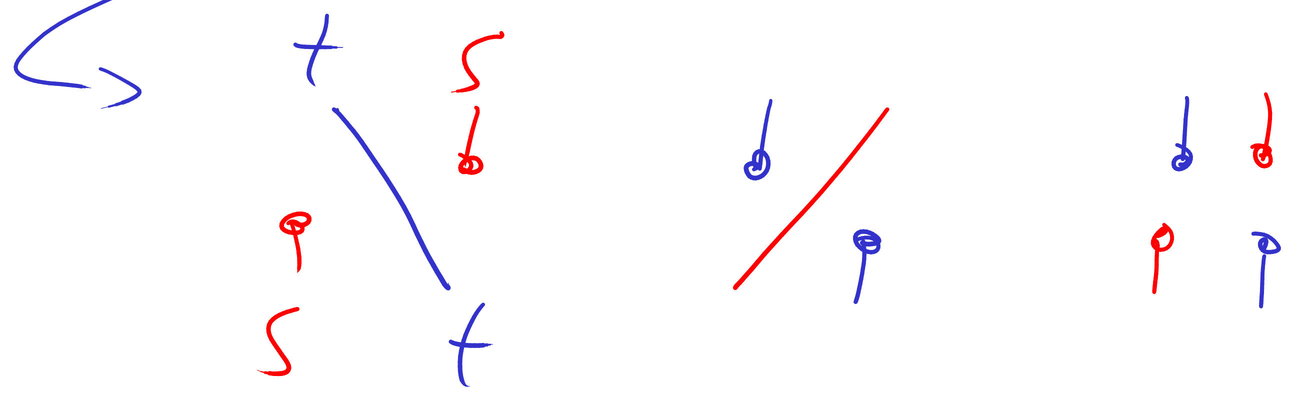
$\in R\langle \text{tree} \rangle$ as left R-module

$\Rightarrow$ Calculus is actually pretty easy

<u>Example</u> $\mathrm{End}(st)$, $\mathrm{Hom}(st, ts)$



all other diagrams will be in the R-span of these



all diagrams will be now in the R-span of these

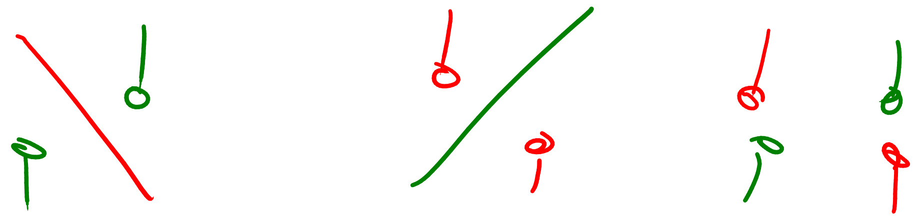
$\Rightarrow \dim \mathrm{Hom}_{D_{BS}(\mathrm{univ})}(st, ts)$

$= 3$-dim over R (not over $\mathbb{C}_{d}^{\vee}$)

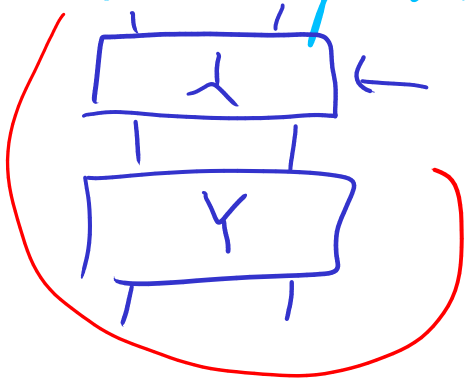
which is fine for $\underset{s}{\circ} \overset{\infty}{\text{———}} \underset{t}{\circ}$

but in $\underset{s}{\circ} \overset{4}{\text{———}} \underset{t}{\circ}$ $\quad st = ts$

$D_4 \hookleftarrow \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle 1, s, t, st \rangle$



=> none of these diagrams is an idempotent in the sense that



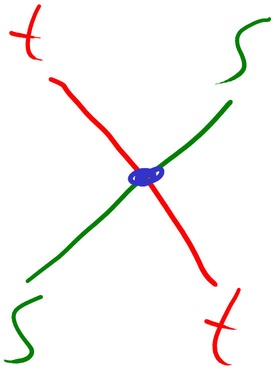
← reversed

no idempotents

=> $B_s B_t \simeq B_t B_s$ can not hold in D_{BS} (uni)!

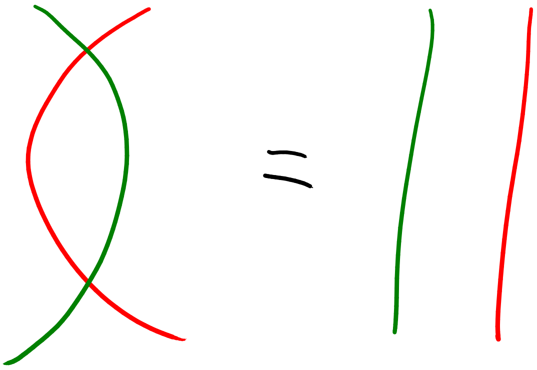
But it should hold in $D_{BS}(\underset{s}{\circ} \overset{2}{\text{———}} \underset{t}{\circ})$.

Trick: Invent a new diagram being the corresponding isomorphism!

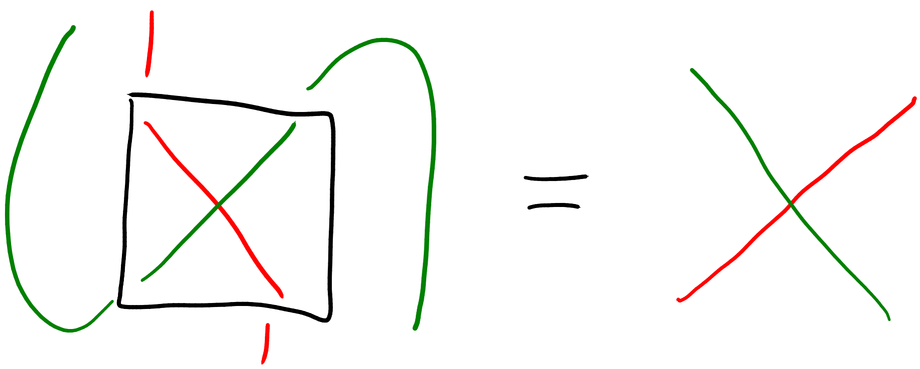


4-valent vertex

Relations needed:

-  plus color inversion

- a few more relations

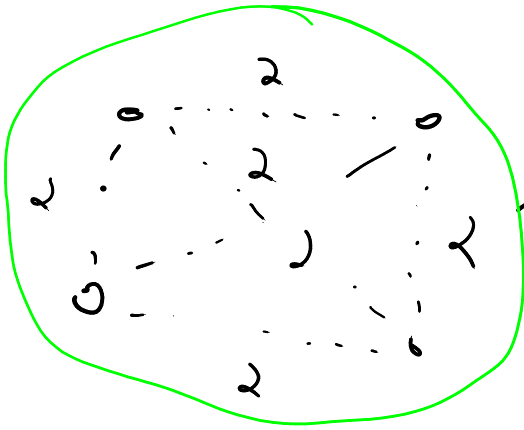
 cyclicity

=> the calculus is still topological?

$\rightsquigarrow$ cat. $\mathcal{D}_{BS}(\mathbb{Z}/2\mathbb{Z}\text{'s})$

<u>Theorem:</u>

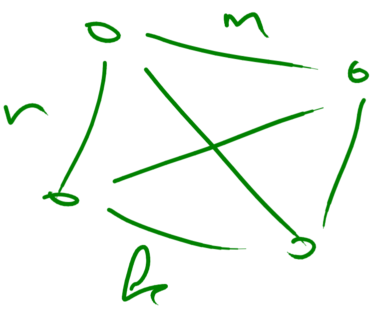
$\mathcal{D}_{BS}(\mathbb{Z}/2\mathbb{Z}\text{'s}) \simeq \mathbb{BS}Bim(W)$

$W =$  $\simeq (\mathbb{Z}/2\mathbb{Z})^{k}$

similarly for $SBim$ $st=ts$ $\forall$ colors

This generalizes as follows

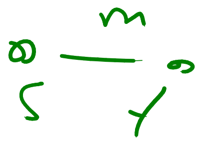
catch here: $\overset{m}{\underset{s\quad t}{\circ\!-\!\circ}}$ only

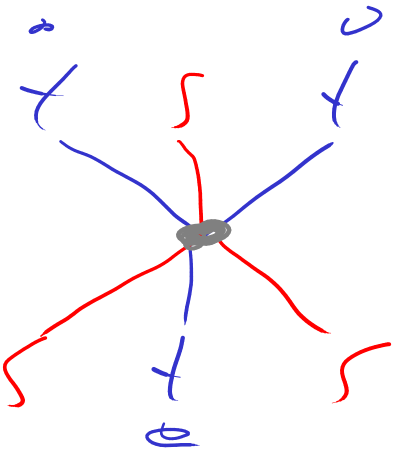


Almost the complete calculus

$m = 3$

$2 \cdot 3$

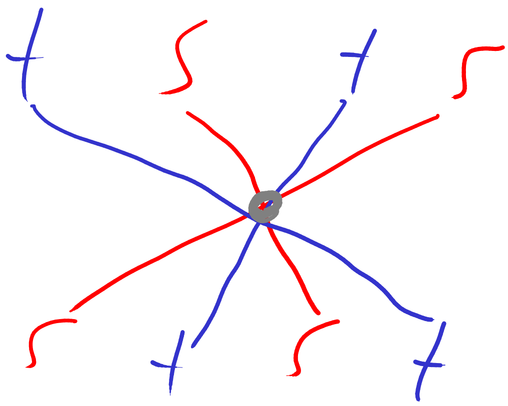




6-valent vertex

$m = 4$

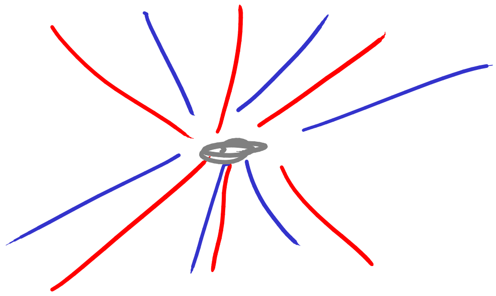
$8 = 2 \cdot 4$



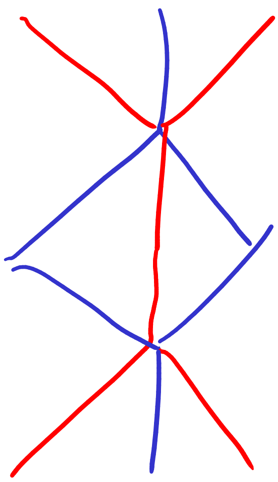
8-valent vertex

$\rightsquigarrow$ $2m$ valent vertex

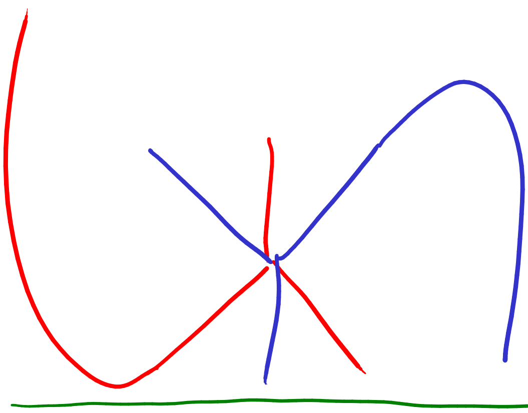
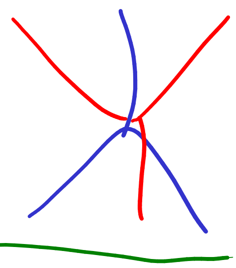
$D_{BS}\left(\underset{s}{\circ} \overset{m}{\text{———}} \underset{t}{\circ}\right)$ same as before + (only 2m-valent) as generators

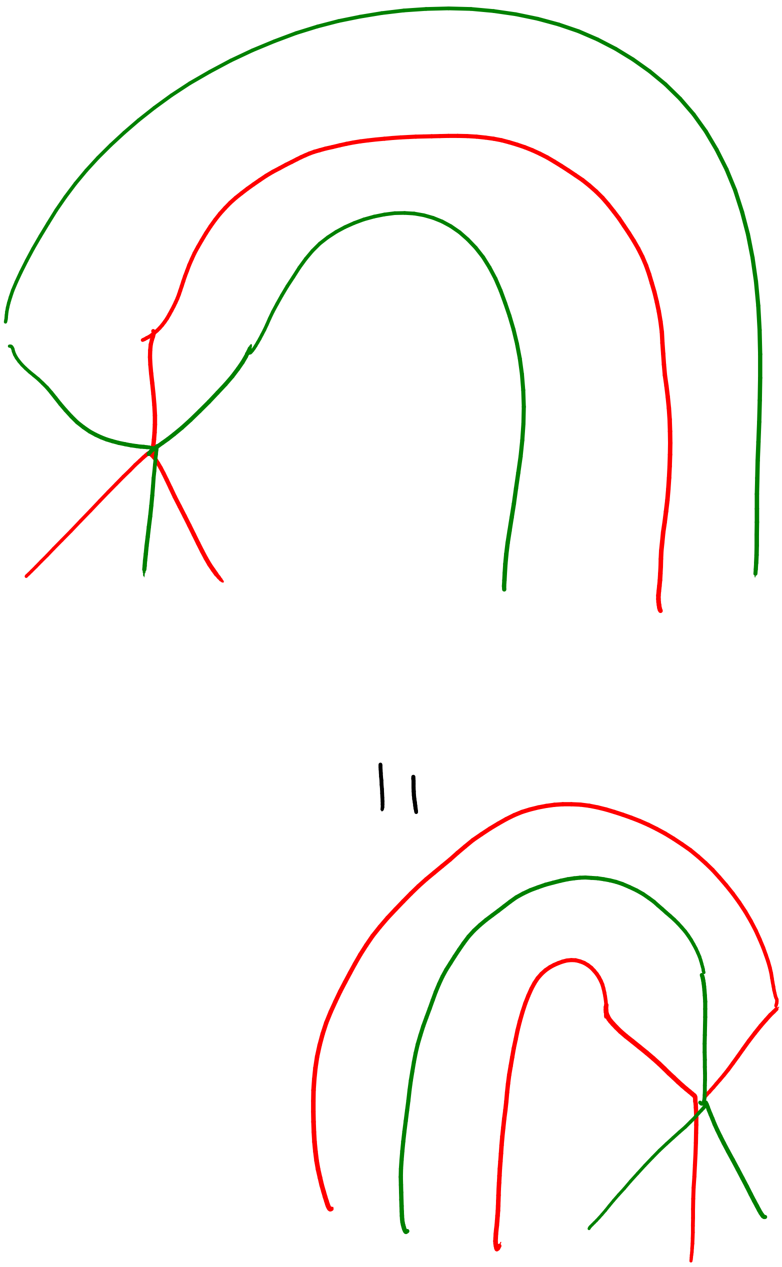


\+ relation as before

\+ 

$\rightsquigarrow$ idempotent in $\mathrm{End}(B_s B_t B_s)$ which splits of B_{sts}

\+  = 

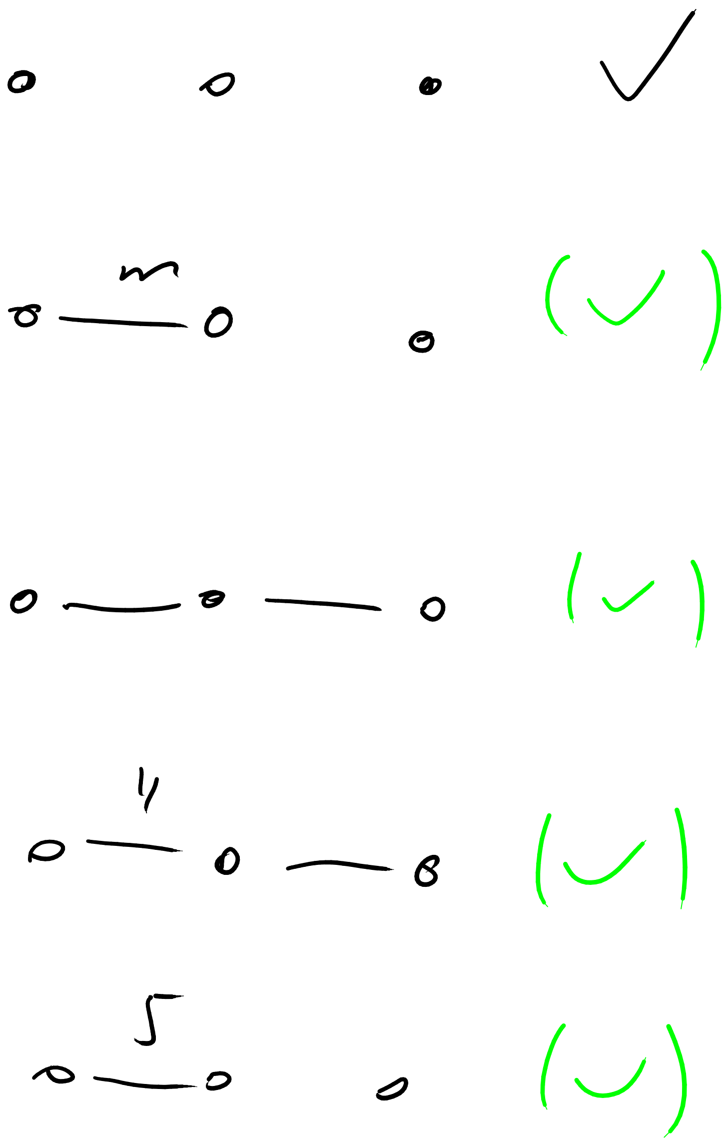


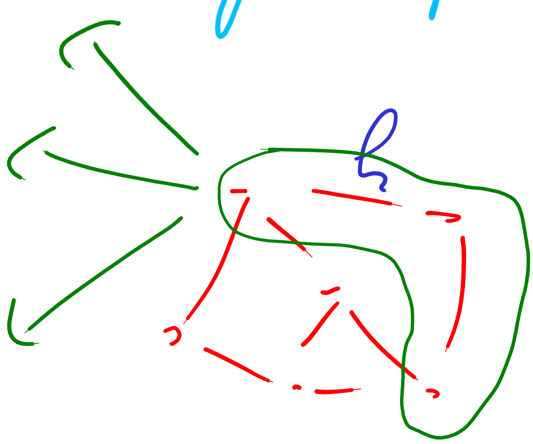
etc.

Thank you!

It turns out that the missing relation only involve 3-cycles

Even better only for the rank 3 finite coxeter groups





$\Rightarrow$ calculus is topological in nature

+ "harmless relations involving $2m$-valent vertex"

Theorem:

$$D_{BS}(\circ \overset{m}{—} \circ) \simeq \mathbb{B}S\mathbb{B}im(\circ \overset{m}{—} \circ)$$

Similarly for $\mathbb{S}\mathbb{B}im$

$\Rightarrow$ Much easier than working in $\mathbb{S}\mathbb{B}im$ itself

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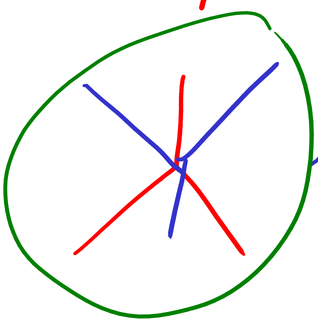
Example: $m=3$



Isos for braid relation

$\rightsquigarrow$ Isomorphism

$$\boxed{\begin{matrix} B_s B_t B_s \\ \simeq \circlearrowleft \\ B_t B_s B_t \end{matrix}} \quad \begin{matrix} \oplus B_t \\ \wr\wr \\ \oplus B_s \end{matrix}$$

braid relation in terms of B_s, B_t