The diagrammatic theory II - Frobenius extensions

Goal: "The one-color Soergel calculus"

Next: "The two-color Soergel calculus"

What does this mean?

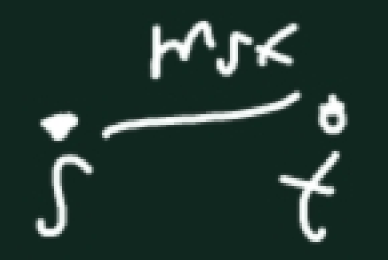
$W = (W, S) \rightsquigarrow$

"Worst case: Three-colors"

One-color

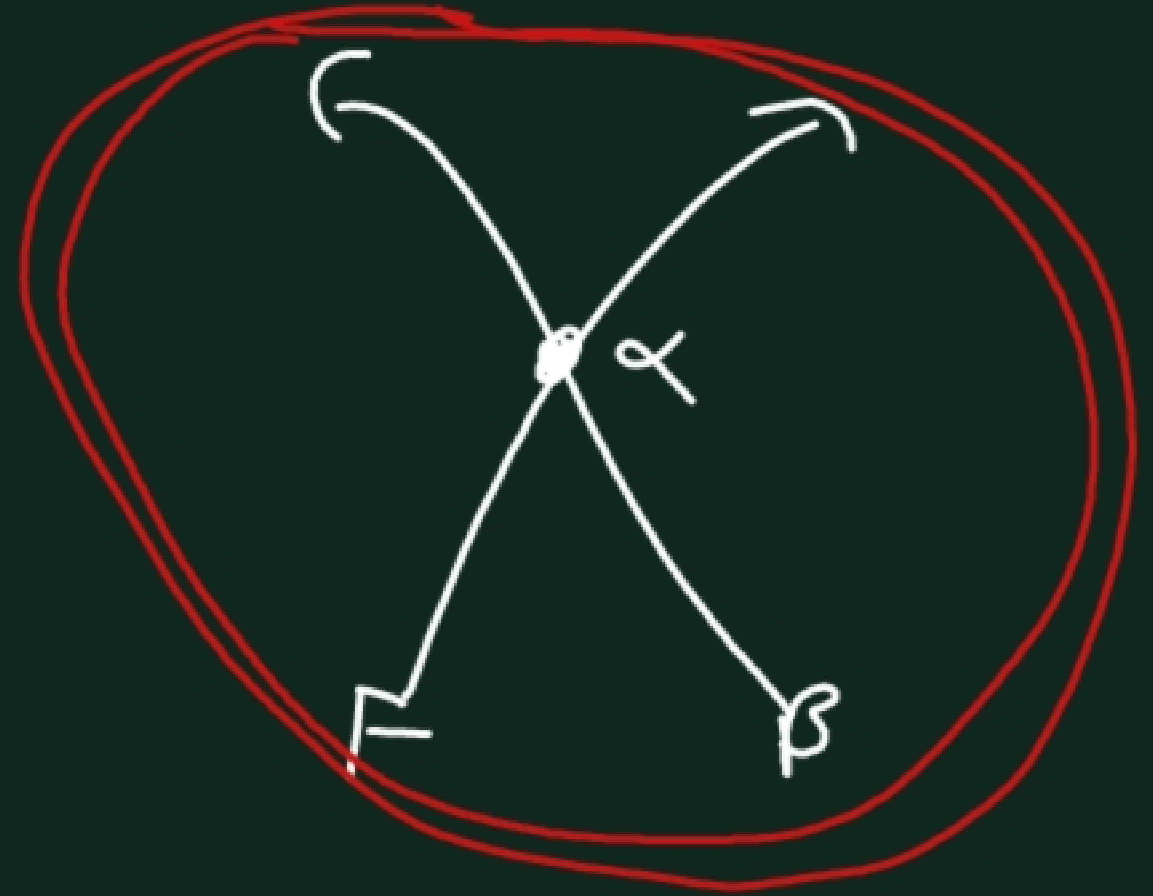
s

Two-color



Recap: Diagrams for 2-cats

2-cat		Monoidal cats 2-cat with one object	
Objects $A, B, \ldots \rightsquigarrow$	faces "2-cells"	/	/
Morphisms $F, G, \ldots \rightsquigarrow$	lines "1-cells"	Objects	lines
2-Morphisms $\alpha, \beta, \ldots$	points "0-cells"	Morphisms	points



Use these for the remainder

Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ a monoidal (strict) category

An algebra $A \in \mathcal{C}$ is an object A together with
(A, μ, c)

two morphisms $\mu : A \otimes A \longrightarrow A$, $c : \mathbb{1} \longrightarrow A$

such that:

$(ab)c = a(bc)$

Associativity holds

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "id \otimes \mu"] \arrow[d, "\mu \otimes id"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

Commutes

Unitality $1a = a = a1$

$$\begin{tikzcd} \mathbb{1} \otimes A \arrow[r, "c \otimes id"] \arrow[dr, "="'] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{1} \arrow[l, "id \otimes c"'] \arrow[dl, "="] \\ & A & \end{tikzcd}$$

Example: An algebra in Vect is an algebra in the usual sense

For example $\mathbb{C}[X]/(X^2) \overset{VS}{=} \mathbb{C}\{1, X\}$

is an algebra with

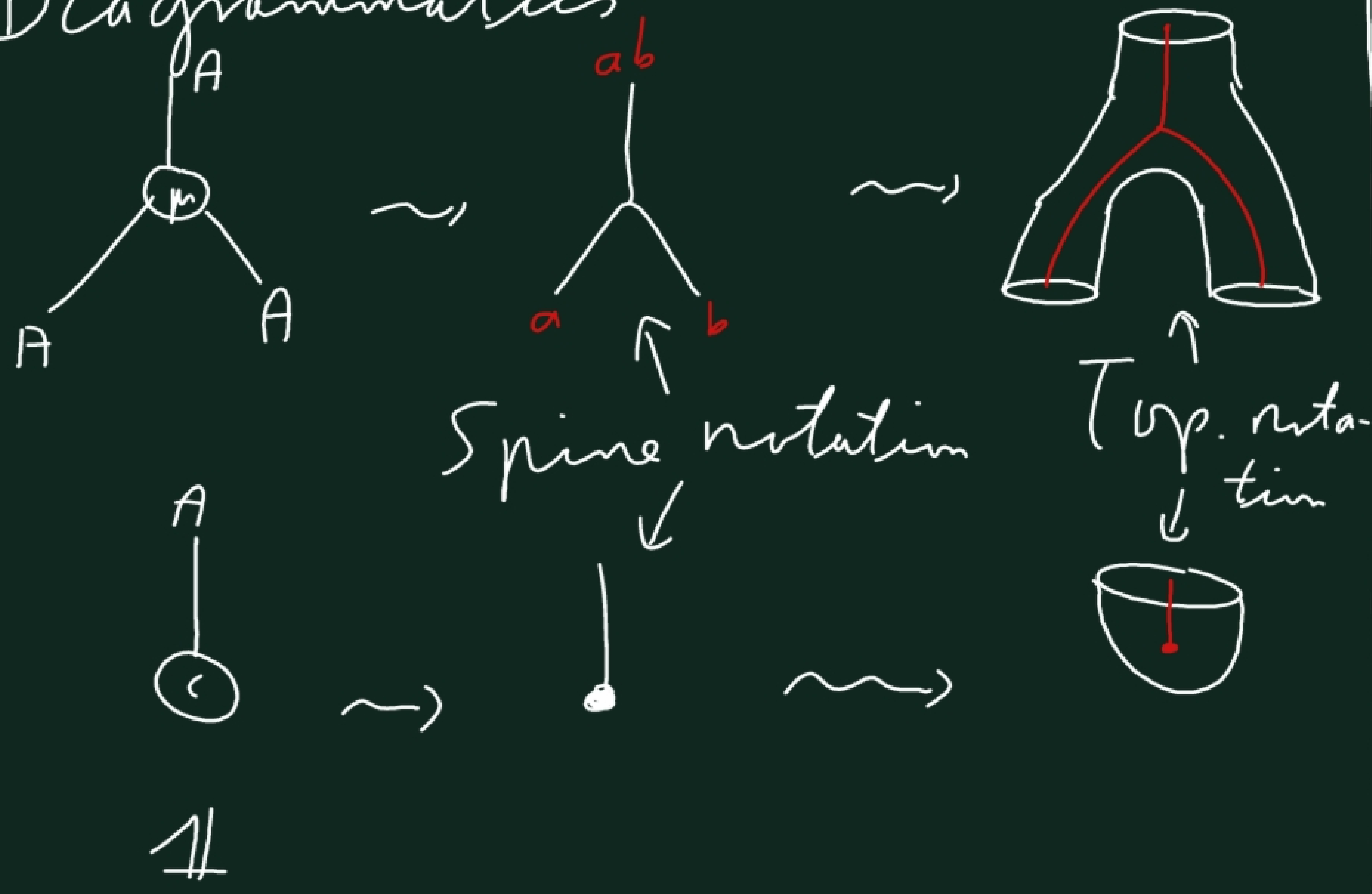
$\mu: \mathbb{C}[X]/(X^2) \otimes \mathbb{C}[X]/(X^2) \to \mathbb{C}[X]/(X^2)$

$\mu(a \otimes b) = ab$

$\iota: \mathbb{C} \to \mathbb{C}[X]/(X^2)$

$1 \longmapsto 1$

Diagrammatics:



Asso:

Unitality:

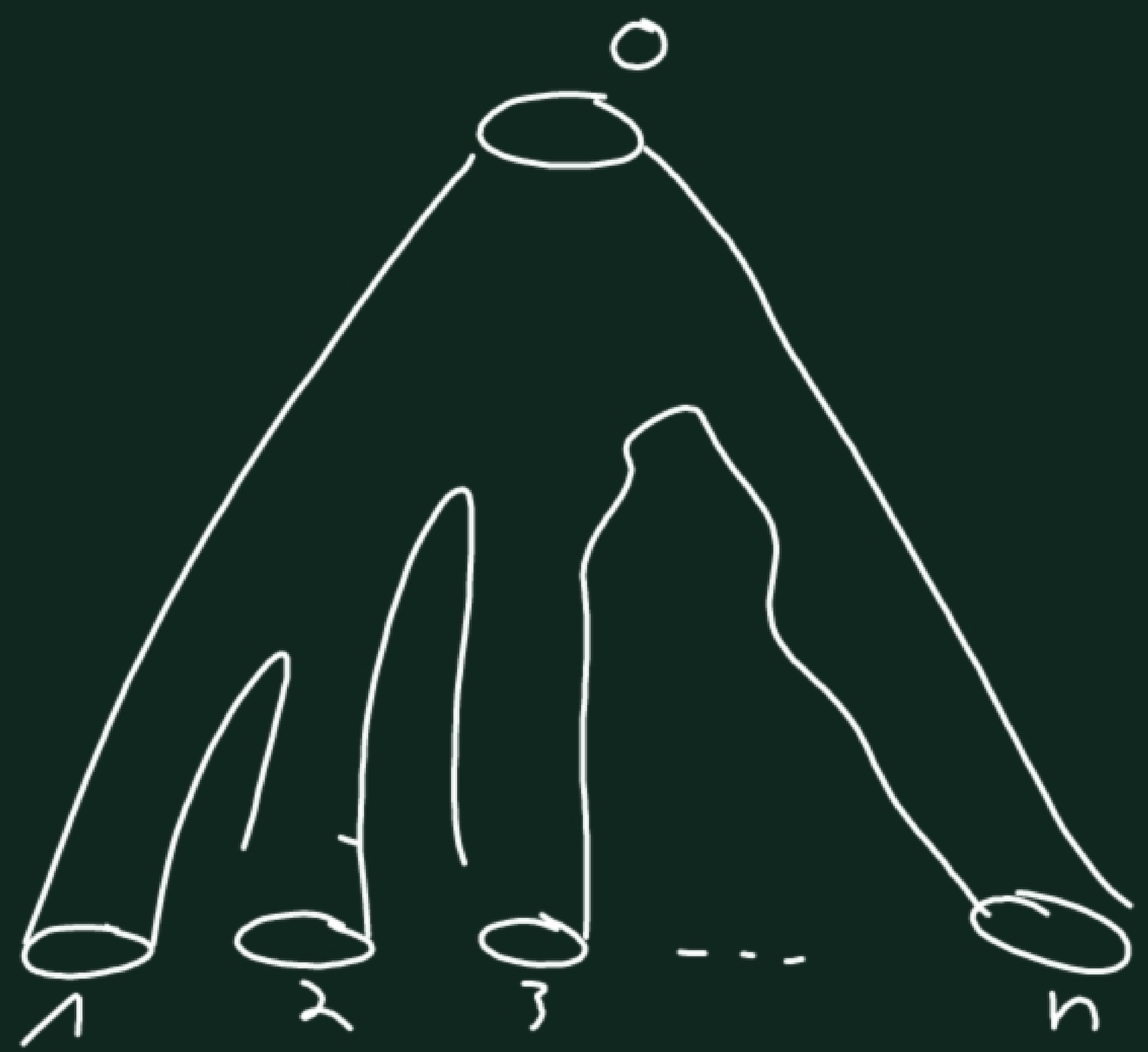
Upshot: Classical
Asso + Unit are topological

"All ways of bracketing are equal"

implies

$$(ab)c = a(bc)$$

$$(ab)(cd) = a(b(cd))$$



$\rightsquigarrow$ $n+1$ - punctured sphere

$\Rightarrow$ "real" associativity

algebra

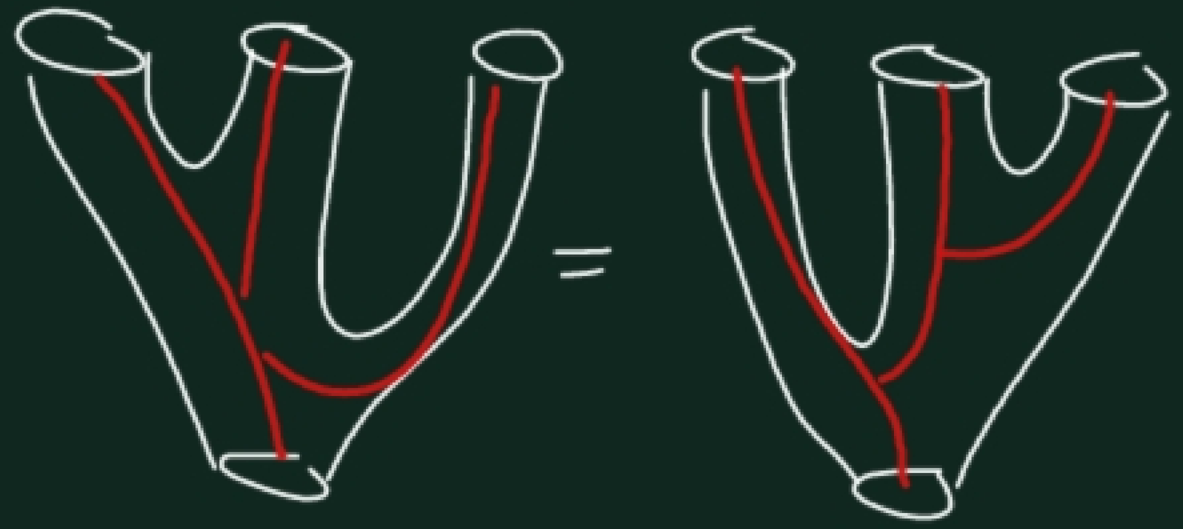
Δ

ε counit

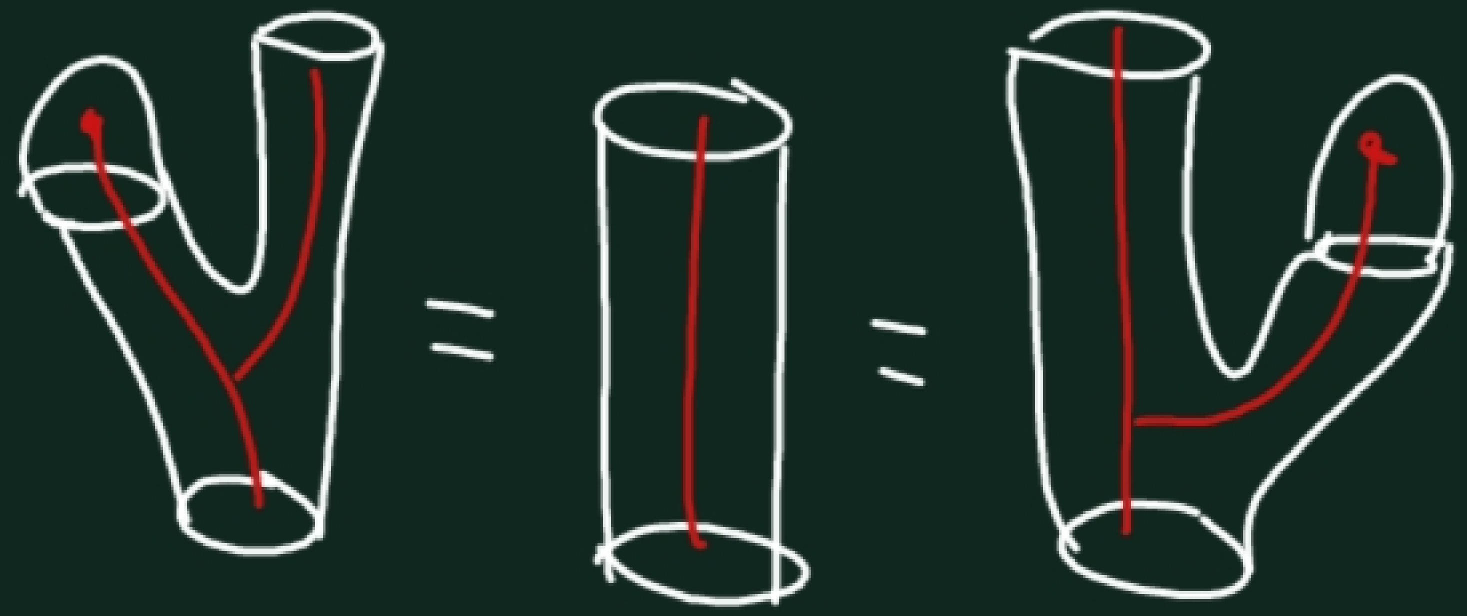
comultiplication

$\rightsquigarrow$

coassoc



counit



Example: Coalgebras in Vect are just classical coalgebras, e.g.

$$\mathbb{C}[X]/(X^2) = \mathbb{C}\{1, X\}$$ is a coalgebra with

$$\Delta : A \to A \otimes A, \quad \begin{cases} 1 \longmapsto 1 \otimes X + X \otimes 1 \\ X \longmapsto X \otimes X \end{cases}$$

$$\varepsilon : A \to \mathbb{1} = \mathbb{C}, \quad \begin{cases} 1 \longmapsto 0 \\ X \longmapsto 1 \end{cases}$$

Key Question: Do we have all "topological" relations using (co) algebra?

Answer: We need one more relation:

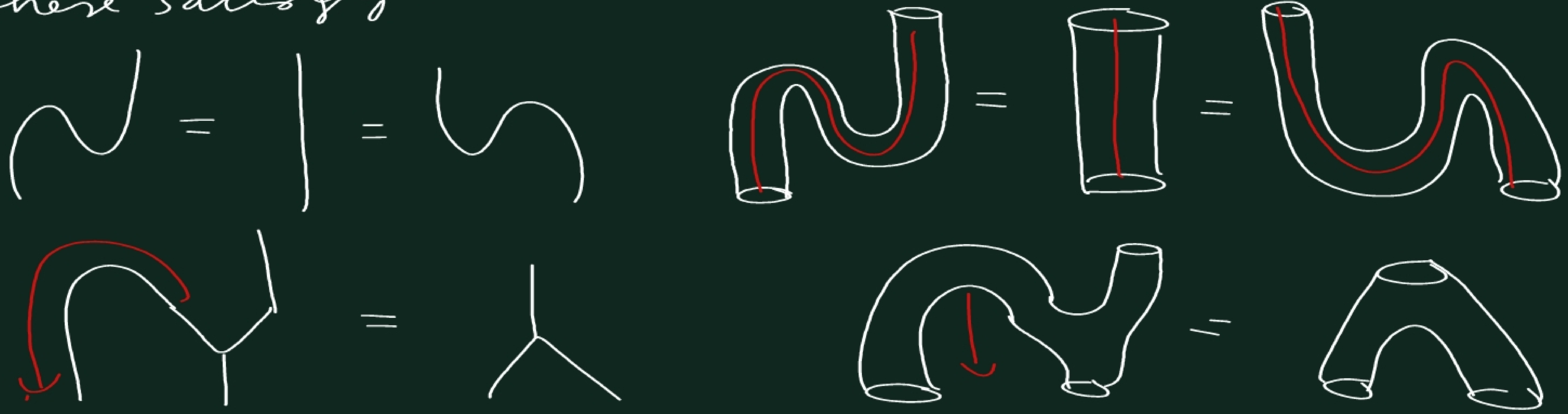
Frobenius relation

A Frobenius algebra $F = (F, \mu, \iota, \Delta, \varepsilon)$ in $\mathcal{C}$ is a algebra + a coalgebra satisfying Frobenius

Example: $\mathbb{C}[x]/(x^2)$ is a Frobenius algebra in Vect

Say we have F Frobenius

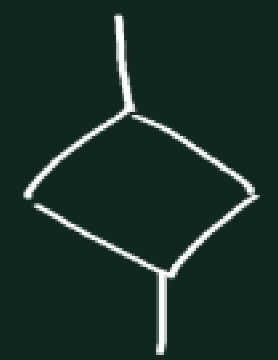
These satisfy:



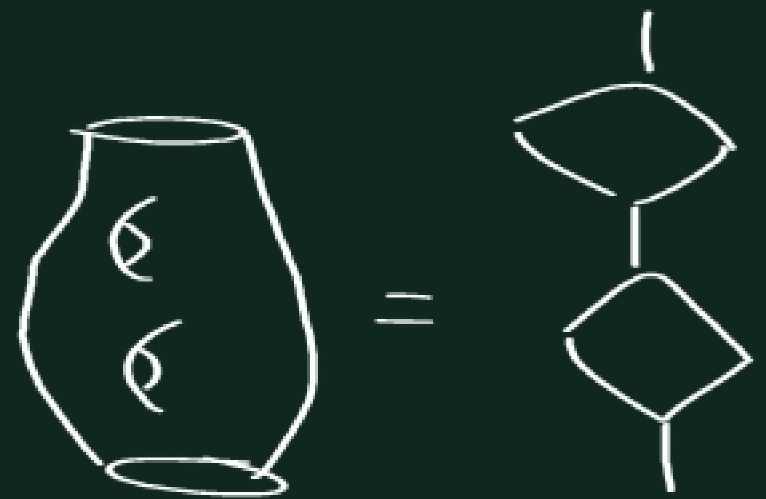
+ etc.

Proposition 8.6 All planar diagrams made from  represent equal morphisms if their diagrams are planar isotopic.

Example: $\mathbb{C}[X]/(X^2)$ 

 $= \mu\circ\Delta :$ $\begin{cases} 1 \xmapsto{\Delta} 1\otimes X + X\otimes 1 \xmapsto{\mu} 2X \\ X \xmapsto{\Delta} X\otimes X \xmapsto{\mu} X^2 = 0 \end{cases}$

"handle operator"

 $= \begin{cases} 1 \longmapsto 2X \longmapsto 0 \\ X \longmapsto 0 \end{cases} = 0$

 $: 1 \longmapsto \overset{\overset{0}{\shortparallel}}{\varepsilon(1)}\otimes X + X\otimes \overset{\overset{0}{\shortparallel}}{\varepsilon(1)} \longmapsto 0$

Draw some morphism made of 

$\cong$ 

genus 8

$\Rightarrow$ morphism is 0

How to produce Frobenius algebras?

A Frobenius extension is the following data:

- A, B two commutative rings
- An inclusion $i : A \longrightarrow B$ $\Rightarrow$ B is an A-module
- A A-linear map $\delta : B \longrightarrow A$ $\leftarrow$ called Frobenius trace

Such that: $\{b_i\}_{i \in I}$, $\{b_i^*\}_{i \in I}$ $|I| < \infty$

- $\exists$ bases of B as an A-module
- The are dual $\delta(b_i b_j^*) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Lemma: If you have the above, then we have four A-bimodule maps

$i : A \longrightarrow B$ $\qquad m : B \otimes_A B \longrightarrow B$

$\delta : B \longrightarrow A$ $\qquad \Delta : B \longrightarrow B \otimes_A B$

making B a Frobenius object in A-Bim-A

Examples: – $A = \mathbb{K}$, B = Frobenius $\Rightarrow$ Frobenius extension

– $B = R = \mathbb{R}[\alpha_s \mid s \in S]$ $\qquad A = \underset{\substack{\uparrow \\ s\text{-sym. polys}}}{\underline{R^s}}$

$i : R^s \hookrightarrow R$ inclusion

$\delta = \delta_s : R^s \longrightarrow R$ Demazure

$m : R \otimes_{R^s} R \xrightarrow{\perp} R \qquad m(f \otimes g) = fg \qquad \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$

$\Delta : R \xrightarrow{\top} \underbrace{R \otimes_{R^s} R}_{B_s} \qquad \Delta(1) = \sum b_i \otimes b_i^*$

$\Delta(1) = 1 \otimes X + X \otimes 1$

$\mathbb{C}[x]/(x^2) \qquad b_i = \{1, X\} \quad b_i^* = \{X, 1\}$

– $A = R$,

$B = B_s = R \otimes_{R^s} R$

$\iota : R \xrightarrow{\top} R \otimes_{R^s} R, \quad 1 \longmapsto 1 \otimes 1$

$\delta : B_s \xrightarrow{\perp} R \qquad f \otimes g \longmapsto fg$

$\mu : B_s \otimes_R B_s \xrightarrow{\curlywedge} B_s \qquad f \otimes g \otimes h \longrightarrow \delta_s(g)\, f \otimes g$

$\Delta : B_s \xrightarrow{Y} B_s \otimes_R B_s \qquad f \otimes g \longmapsto f \otimes 1 \otimes g$

Such that B_s is a Frobenius object in R-Bim, in particular also in $\mathbb{BS}$Bim and $\mathbb{S}$Bim

$\Rightarrow$ Topological relations for B_s

$(B \otimes_A \cancel{B) \otimes_B} \otimes (B \otimes_A B)$

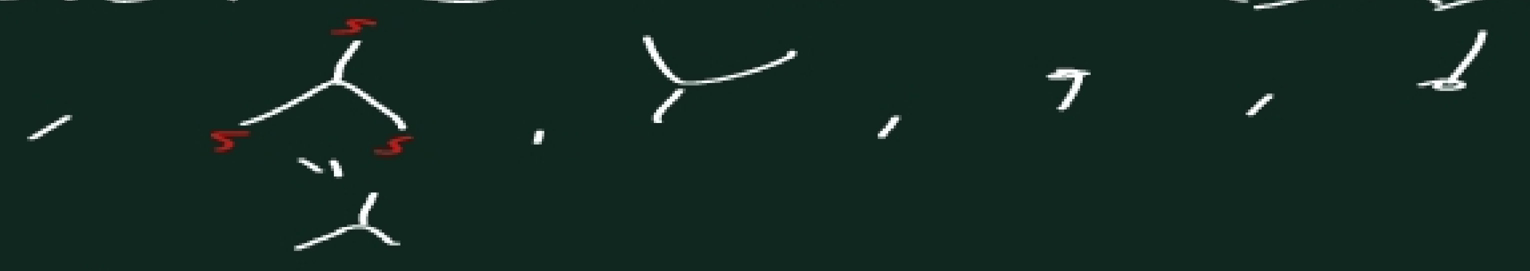
$\rightleftarrows B \otimes_A B \otimes_A B$ $\quad \downarrow\delta \ \uparrow i$

$\rightleftarrows B \cancel{\otimes_A A} \otimes_A B$

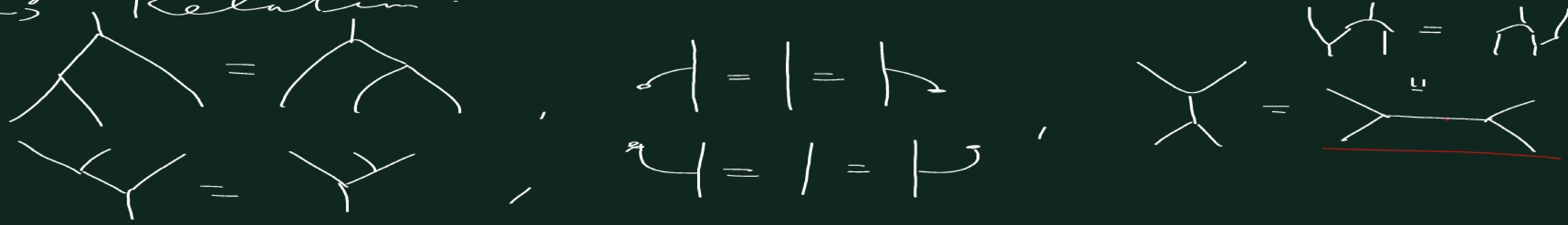
$\rightleftarrows B \otimes_A B$

Question: What is now the 1-color calculus?

– $\langle R, B_s \rangle$,



Towards Relation:

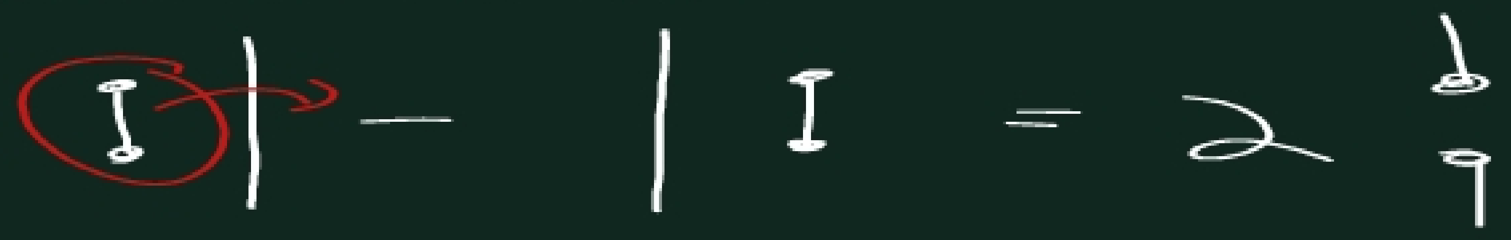


– Two extra relations

Handle relation

"Frobenius"

Slide relation



"a honest Soergel relation"

• s Coxeter diagram

$\Rightarrow$ cat $\mathcal{H}_{BS} = \mathcal{H}_{BS}$ (type A_1) $= (\mathcal{H}_{BS}, \mathbb{1} = R, \otimes = \otimes_R)$

generated by $B_s \leftrightarrow |_s$ objects

 morphisms

subject to the above relation

$\mathcal{H}_{BS}$ is called the 1-color Soergel calculus

Example:

$\stackrel{?}{=} 0$

because of Frobenius

$+$  $= 0$

Wrap-up:

Theorem $\exists\, F : \mathcal{H}_{BS} \longrightarrow \mathbb{BS}\mathrm{Bim}$ (type A_1)

- F is a $\otimes$-functor
- F is essentially surjective
- F fully faithful

$\Rightarrow$ F is an equivalence

$\oplus, \oplus$

Thus, $\mathcal{H}_{BS}$ is a diagrammatic model of $\mathbb{BS}\mathrm{Bim}$

" " " " $\mathbb{S}\mathrm{Bim}$