The classical theory III – Kazhdan–Lusztig theory (KL)

(W, S) Coxeter system

Group ring $\mathbb{Z}[W]$

Basis $\{x \in W\}$

Mult: $w \cdot w' = ww'$

Relations $\quad W = \langle s \in S \rangle$

- $s^2 = 1$
- braid rel. $\underbrace{sts\ldots}_{m_{st}} = \underbrace{tst\ldots}_{m_{st}} \leftrightarrow (st)^{m_{st}} = 1$

Hecke algebra $H = H(W, S)$

The $\mathbb{Z}[v, v^{-1}]$-algebra generated by $\{\delta_x \mid x \in W\}$

Relations: $\to$ quantum term

- $\delta_s^2 = (v^{-1} - v)\delta_s + 1$ (Quadratic relation)
- $\underbrace{\delta_s \delta_t \delta_s \ldots}_{m_{st}} = \underbrace{\delta_t \delta_s \delta_t \ldots}_{m_{st}}$ (Braid relation)

- $\mathbb{Z}[v, v^{-1}]$

$\hookrightarrow$ expression of the form $\sum a_k v^k \quad k \in \mathbb{Z}$, $a_k \in \mathbb{Z}$

- Not allowed: $\frac{1}{1-v}, \frac{1}{v+2}, \ldots$

$\Rightarrow$ For every ring R and every choice $\varepsilon \in R^*$ (invertible)

We get specialization

$$\mathbb{Z}[v, v^{-1}] \longrightarrow R$$
$$1 \longmapsto 1$$
$$v \longmapsto \varepsilon$$

For short $v \mapsto \varepsilon$

e.g. – $R = \mathbb{C}$, $\varepsilon = 1$ „classical"
– $R = \mathbb{Z}$, $\varepsilon = 1$
– $R = \mathbb{C}$, $\varepsilon = \pi/e$

$\Rightarrow$ Consider $H \otimes_{\mathbb{Z}[v, v^{-1}]} R$

Idea $H \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Z} \cong \mathbb{Z}[W]$ ($v \mapsto 1$)

"quantization"

We will see in action that we want v and v^{-1}!

$x \in W$ fix rex $\underline{x} = s_1 \cdots s_m$

$v \mapsto 1$

Define $\delta_{\underline{x}} = \delta_{s_1} \cdots \delta_{s_m}$

Consider $\{\delta_x \mid x \in W\} \subset H$

$\{x \mid x \in W\} \subset \mathbb{Z}[W]$ basis

Theorem: The set $\{\delta_x \mid x \in W\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of H. ← Standard basis

$\Rightarrow$ H is a flat deformation of W

How could we prove this?

- linear independence is a bit tricky ("construct faith rep")
- Spanning? Let me take a detour!

General philosophy:

- "Everything lives in W"

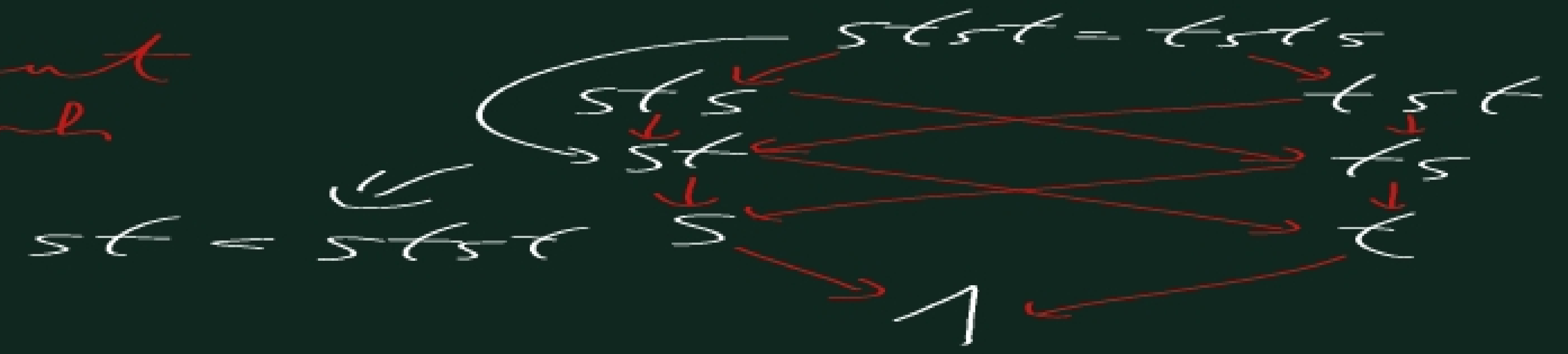
Explicitly: There is the Bruhat order on W defined by: $x \leq y \iff \exists$ rex $\underline{x}$ of x and rex $\underline{y}$ of y st. $\underline{x}$ is a subexpression of $\underline{y}$

$\underline{y} = s_1 \cdots s_m$
$\underline{e} = 0010 \cdots 1$

$\underline{y} = s\,t\,s\,t\,u\,t\,u\,t$
$\underline{e} = 0\,1\,0\,1\,1\,1\,0\,1$
$\underline{x} = \;\; t \;\; t\,u\,t \;\; t$
$\underline{x}$ is a subexpression of $\underline{y}$

Example $s \overset{4}{—} t$ $\quad W = \{1, s, t, st, ts, sts, tst, stst = tsts\}$ ($stst = tsts = w_0$)

Bruhat graph



Fact: $\forall x \in W$ the set $\{y \in W \mid y < x\}$ is finite

Note:
$$\delta_x \delta_s = \begin{cases} \delta_{xs} & x < xs \\ \delta_{xs} + (v^{-1} - v)\,\delta_x & x > xs \end{cases}$$

$(v^{-1} - v) \xrightarrow{v \mapsto 1} 0$

$$xs = xs$$

$\Rightarrow$ Spanning!

<u>Definition</u> (KL)

A KL basis $\{b_x \mid x \in W\} \subset H$ is such a set s.t.

1) $\overline{b_x} = b_x$ (self-duality)

2) $b_x = \delta_x + \sum_{y<x} h_{y,x} \delta_y$ KL polynomial $h_{y,x} \in v\mathbb{Z}[v]$

(Δ - base change)

bar involution

There exists involution $\bar{\ } : H \to H$

not self-dual $\overline{\delta_x} = \delta_{x^{-1}}^{-1}$ $b_x \neq \delta_x$ in general!

$\bar v = v^{-1}$

Extend $\overline{ab} = \bar a \bar b$

How to think about 2?

$$\begin{array}{cc} & b_x \\ \delta_x & \begin{pmatrix} 1 & \cdots & v\mathbb{Z}[v] \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \end{array} \quad h_{y,x}$$

- Q1 Is such a KL basis unique? YES!
- Q2 Does it exist? YES!

Example: $s \overset{3}{\text{---}} t$ $W = \{1, s, t, st, ts, sts = tst\}$ ($sts = w_0$)

SB: $\{\delta_1, \delta_s, \delta_t, \delta_{st}, \delta_{ts}, \delta_{w_0}\}$

Want KL: $\{b_1, b_s, b_t, b_{st}, b_{ts}, b_{w_0}\}$

Answer: Base-change

$$\begin{array}{cc} & b \\ \delta & \begin{pmatrix} 1 & v & v & v^2 & v^2 & v^3 \\ & 1 & 0 & v & 0 & v^2 \\ & & 1 & v & v & v^2 \\ & & & 1 & v & v \\ & 0 & & & 1 & v \\ & & & & & 1 \end{pmatrix} \end{array} \begin{array}{l} 1 \\ s \\ t \\ ts \\ st \\ w_0 \end{array} \quad \to v\mathbb{Z}[v]$$

$b_{w_0} = \delta_{w_0} + v\,\delta_{st} + v\,\delta_{ts} + v^2 \delta_s + v^2 \delta_t + v^3 \delta_1$

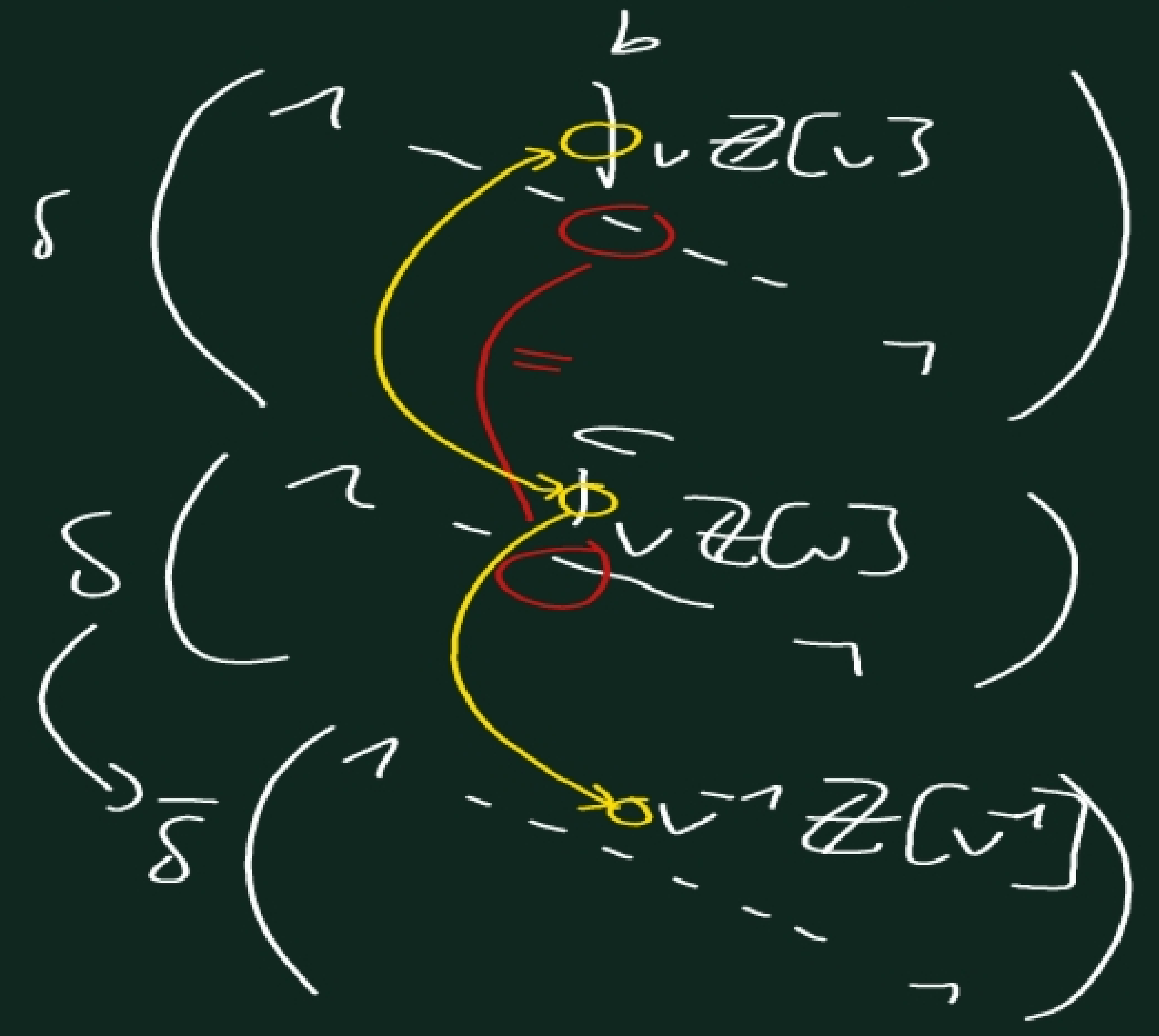
In general a miracle will happen: $h_{y,x}$ are actually in $v\mathbb{N}[v]$

No negative numbers

Uniqueness?

Assume $\{b_x \mid x \in W\}$ + $\{c_x \mid x \in W\}$ are two KL basis.

Consider $x \in W$ where $b_x \neq c_x$



$$b_x - c_x = \sum_{y<x} (h_{y,x} - h'_{y,x})\,\delta_y$$

$$\|$$

$$\overline{b_x - c_x} = \sum_{y<x} \overline{(h_{y,x} - h'_{y,x})}\;\overline{\delta_y}$$

Lemma: $\overline{\delta_y} = \delta_y$ + lower order terms

$\Rightarrow \exists\, y$ such that $\underset{v^{-1}\mathbb{Z}[v^{-1}]}{\overline{(h_{y,x} - h'_{y,x})}} = \underset{v\mathbb{Z}[v]}{(h_{y,x} - h'_{y,x})}$

$$v\mathbb{Z}[v] \cap v^{-1}\mathbb{Z}[v^{-1}] = \emptyset \;\rightarrow\; \lightning$$

Note: I needed v^{-1} !

Existence?

Back to geometry of W

Example: $s \overset{3}{—} t$ Coxeter complex

$\rightsquigarrow$ Representing $\{\delta_x \mid x \in W\}$

Want to write

$\rightsquigarrow v\,\delta_s + (v^{-1}+1)\,\delta_{ts} + (v^3 - v)\,\delta_{w_0}$

$f \in \mathbb{Z}[v, v^{-1}]$

KL game:

$\longrightarrow$

v, v^{-1}

if exp. gets bigger "Up"

if it gets smaller "Down"

Define $\{b_x \mid x \in W\}$ inductively

$b_1 =$  $= \delta_1$

$$\delta_x b_s = \begin{cases} \delta_{xs} + v\,\delta_x & \text{if } xs > x \quad \text{Up} \\ \delta_{xs} + v^{-1}\delta_x & \text{if } xs < x \quad \text{Down} \end{cases}$$

$\longrightarrow$ Extract the biggest self-dual term?

$b_s = 1\cdot\delta_s + v\,\delta_1$

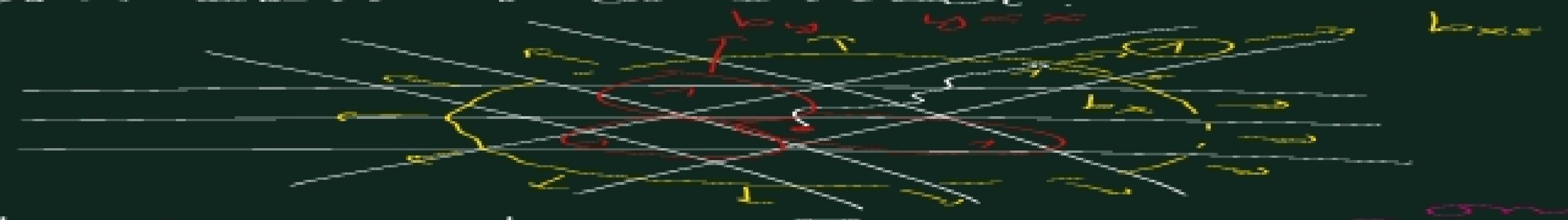
$b_t = 1\cdot\delta_t + v\,\delta_1$

Coxeter complex

Bruhat order

Theorem (KL): This always works and constructs the KL basis.

In other words: If you find something that is not self dual, then $\exists$  $= b_y$, $y < x$ which you can subtract.



Even better: $b_x = \delta_x +$ friends

only have coeff in $\mathbb{N}[v]$

$\rightsquigarrow$ something is counted?

KL positivity conjecture (1979)

Soergel–Elias–Williamson gave a proof in 2011

Goal: Understand what is counted by $h_{y,x}$-poly's