Tubbenhauer, 15.Apr.2019

"Universal enveloping algebra II – the Cartan subalgebra"

Main goal of this lecture:
Identify the center of $U(\mathfrak{g})$

Recall: $U(\mathfrak{g})$ is the algebra gen by e, f, h modulo:

$$\boxed{\begin{aligned} ef - fe &= h \\ he - eh &= 2e \\ hf - fh &= -2f \end{aligned}}$$

EF Relation
EH Relation
FH Relation

Fact (PBW): $U(\mathfrak{g})$ is an

∞-dim algebra with basis $\{f^i h^j e^k \mid i,j,k \in \mathbb{N}\}$

"Proof": Rewrite recursively words eg:

Choose an order:

① $ef - \boxed{fe} = h$ $\quad ef = fe + h$

② $\boxed{he} - eh = 2e$ $\quad eh = he - 2e$

③ $hf - \boxed{fh} = -2e$ $\quad hf = fh - 2e$

Now: - Perform ① until no expression of the form ef remain

- Perform ② until no eh remain
- Perform ③ until no hf remain
- repeat.

Such an algorithm always works if one has a "nice" filtration!

Fix the degree: $\deg(e) = \deg(f) = \deg(h) = 1$

$\rightsquigarrow$ extend additively to words.

Example:

$$\underbrace{ef}_{\deg 2} - \underbrace{fe}_{\deg 2} = \underbrace{h}_{\deg 1}$$

$\Rightarrow$ not graded but only filtered, i.e. the degree can not increase formally: $U(\mathfrak{g})^{-1} = 0$

$U(\mathfrak{g})^{i}$ = span of all monomials of degree at most i

$\rightsquigarrow U(\mathfrak{g}) = \bigcup_i U(\mathfrak{g})^i$

The point:

$$U(\mathfrak{g})^i \, U(\mathfrak{g})^j \subset U(\mathfrak{g})^{i+j}$$

Filtration

Example:

$U(\mathfrak{g})^0 = \langle 1 \rangle$

$U(\mathfrak{g})^1 = \langle 1, e, f, h \rangle$

$U(\mathfrak{g})^2 = \langle 1, e, f, h, e^2, f^2, h^2, fe, fh, he \rangle$

etc.

In particular: PBW of at most degree i are a basis of $U(\mathfrak{g})^i$

Crucial: The associated graded is

$$\boxed{\begin{aligned} G(\mathfrak{g})_i &:= U(\mathfrak{g})^i / U_{\mathfrak{g}}^{i-1} \\ G(\mathfrak{g}) &:= \bigoplus G(\mathfrak{g})^i \end{aligned}}$$

Clearly:

$$G(\mathfrak{g})_i \, G(\mathfrak{g})_j \subset G(\mathfrak{g})_{i+j}$$

deg i deg j deg $i+j$

Compare:

$$U(\mathfrak{g})^i \, U(\mathfrak{g})^j \subset U(\mathfrak{g})^{i+j}$$

deg at most i deg at most j deg at most $i+j$

"filtered $\Leftrightarrow$ graded"

$\leq$ $\Leftrightarrow$ $=$

Example:

$G(\mathfrak{g})_0 = \langle 1 \rangle$

$G(\mathfrak{g})_1 = \langle e, f, h \rangle$

$G(\mathfrak{g})_2 = \langle e^2, f^2, h^2, fe, fh, he \rangle$

Clearly: $G(\mathfrak{g})_i$ has a basis of PBW's of degree i

---

Proposition:

$$G(\mathfrak{g}) \simeq \mathbb{C}[e,f,h]$$

Proof: $e \mapsto e$, $f \mapsto f$, $h \mapsto h$ does the trick, if well-defined

(1) $ef - fe = \cancel{h}$

(2) $he - eh = \cancel{2e}$

(3) $hf - fh = -\cancel{2f}$

killed because of lower degree

$\hookrightarrow$ well-defined

Corollary: $U(\mathfrak{g})$ is a domain, i.e. $xy = 0 \Rightarrow x = 0$ or $y = 0$.

Proof: Push the statement to $G(\mathfrak{g}) \cong \mathbb{C}[e, f, h]$

Upshot: Words in e, f, h have leading terms!

$$e f h e = f e h e + \text{lower}$$
$$= \underline{f h e^2} + \text{lower}$$

leading term "polynomial part"

We get this directly by "commuting"

Recall: The Cartan sub-algebra $\mathfrak{h} < \mathfrak{g}$
$$\mathfrak{h} = \langle h \rangle$$
plays an important role since its action is always diagonalizable "weights".

We will now see "why it is so special"

---

(1) $\overset{+1}{e}\overset{-1}{f} - \overset{-1}{f}\overset{+1}{e} = \overset{0}{h}$

(2) $\overset{0}{h}\overset{+1}{e} - \overset{+1}{e}\overset{0}{h} = 2\overset{+1}{e}$

(3) $\overset{0}{h}\overset{-1}{f} - \overset{-1}{f}\overset{0}{h} = -2\overset{-1}{f}$

This is graded if:

$\deg(e) = 1$ $\qquad \deg(f) = -1$

$\deg(k)=0$

$U(g)_i$ = span of monomials of degree i

$\rightsquigarrow$ $U(g) = \bigoplus U(g)_i$

$U(g)_i \, U(g)_j \subset U(g)_{i+j}$

Graded, but not positively graded !

Example: Is a subalgebra

$h \in U(g)_0$, $1 \in U(g)_0$

$c = \underbrace{(h^2+1)^2}_{\deg 0} + 4\underset{-1}{f}\underset{+1}{e}$

$\in U(g)_0$

$\rightsquigarrow$ all of these played an important role in the classification of simples!

Proposition: $U(\mathfrak{g})_0$ is ∞-dimensional, namely $U(\mathfrak{g})_0 \simeq \mathbb{C}[h, c]$

Proof: First, we clearly have $\mathbb{C}[h, c] \subset U(\mathfrak{g})_0$. So it remains to rewrite each $f^i h^j e^k$, $\underbrace{-i+k=0}_{\Rightarrow\, i=k!}$ in terms of h, c.

$\rightsquigarrow$ Simplify things by choosing a different PBW basis

basis: $f^i e^j \ell^n$, with $\in U(g)_0$

$\Leftrightarrow i = j$

Induce on i. $c = (\ell+1)^2 + 4fe$

$i = 0$ ✓

$i = 1$ $fe = \frac{1}{4}(c - (\ell+1)^2)$ ✓

$i > 1$ $f^i e^i = f(f^{i-1} e^{i-1})e$

$= \underbrace{(fe)}_{\in \mathbb{C}[c,\ell]} \underbrace{(f^{i-1} e^{i-1})}_{\in \mathbb{C}[c,\ell]} + \text{Rest}$

because of $i = 1$ induction

$\text{Rest} = f \underbrace{[f^{i-1}, e]}_{\text{can be also treated inductively}} e^{i-1}$

Now the center:

$$Z(g) = \{x \in U(g) \mid xy = yx \ \forall y \in U(g)\}$$

Interesting: Elements for $Z(g)$ act as scalars on simple reps!

Example: $C \in Z(g)$

Question: Is there another element in the center? "No":

Theorem: $Z(g) = \mathbb{C}[C] \subset U(g)$.

Proof: We have seen that $\mathbb{C}[C] \subset Z(g)$

Moreover: $Z(g) \subset U(g)_0$

$= \mathbb{C}[c, h]$

= all elements,
which commute
with h

$\rightsquigarrow$ $x \in Z(g)$
$\Rightarrow$ $x = \sum_{i,j} h^i c^j$
x is central $\Rightarrow [e, x] = 0$
Calculation $[e, x] =$
$e \sum_{i,j} (h^i - (h+2)^i) c^j$
$U(g)$ is a domain $\Rightarrow$
$\sum_{i,j} = 0 \overset{\text{calculation}}{\Longrightarrow} i = 0$ only
possibility.

Morally: h does not commute
with e $\Rightarrow$ $h \notin Z(g)$ and

it can not appear in expression of $x \in Z(g)$

Finally, without proof:

Theorem: - $U(g)$ is free over $U(g)_0$ with basis

$$B_1 = \{1, e, f, e^2, f^2, e^3, f^3, \dots\}$$

- Free over $Z(g)$ with basis

$$\{1, e, e^2, \dots\} \cdot B_1$$

Last modified: 11:50