Chapter 2 : Projective and Injective Representations

Projective representations and injective representations are key concepts in representation theory. A representation P is called projective if the functor $\mathrm{Hom}(P,-)$ maps surjective morphisms to surjective morphisms. A representation I is called injective if the functor $\mathrm{Hom}(-,I)$ maps injective morphisms to injective morphisms. For any representation M there is a projective representation P_0 s.t. there exists a surjective morphism:
$$p_0 : P_0 \twoheadrightarrow M$$
For any representation M there is an injective representation I_0 s.t. there exists an injective morphism:
$$i_0 : M \hookrightarrow I_0$$
If M is not projective itself, then the morphism p_0 above will have a kernel, and we can find another projective P_1 s.t. there exists a surjective morphism p_1 from P_1 to the kernel of p_0.
$$\cdots \longrightarrow P_3 \xrightarrow{p_3} P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0 \qquad \text{(iterating the procedure)}$$
each P_i is a projective representation. Such a sequence is called a projective resolution. Projective resolutions are a way to approximate the representation M by projective representations. Often it is possible to deduce properties of M from a projective resolution of M.
We also have injective resolutions, exact sequences of the form
$$0 \to M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} I_2 \xrightarrow{i_3} I_3 \longrightarrow \cdots$$
each I_i is an injective representation.
For representations of quivers without oriented cycles the situation is simple.
Every quiver representation has a projective resolution of the form
$$0 \to P_1 \to P_0 \to M \to 0$$
and an injective resolution of the form
$$0 \to M \to I_0 \to I_1 \to 0 \qquad \text{(we will see this later)}$$
With this result we can show that every subrepresentation of a projective representation is projective. Categories with this property are called hereditary.
It is easy to write down all indecomposable projective representations of a quiver Q without oriented cycles. There is exactly one indecomposable projective representation $P(i)$ for each vertex $i \in Q_0$ and this representation $P(i)$ is given by the paths in Q starting at the vertex i. There is exactly one indecomposable injective representation $I(i)$ for each vertex $i \in Q_0$ and $I(i)$ is given by the paths ending at the vertex i.

Def 2.1 Let $Q = (Q_0, Q_1, s, t)$ be a quiver, $i,j \in Q_0$. A path c from i to j of length ℓ in Q is a sequence
$$c = (i \mid \alpha_1, \alpha_2, \dots, \alpha_\ell \mid j)$$
with $\alpha_h \in Q_1$ s.t.
$$s(\alpha_1) = i$$
$$s(\alpha_h) = t(\alpha_{h-1}), \quad \text{for } h = 2,3,\dots,\ell$$
$$t(\alpha_\ell) = j$$

Thus a path from i to j is a way to go from vertex i to vertex j in Q, we are only allowed to walk along an arrow in the direction it's pointing.

Exp 2.1 In the quiver

$$\alpha \circlearrowleft 1 \xrightarrow{\beta} 2 \xleftarrow{\gamma} 3$$

we have that $(1|\alpha|1)$, $(1|\alpha,\beta|2)$, $(1|\alpha,\alpha,\beta|2)$ are paths. $(1|\alpha,\beta,\gamma|2)$ is not a path as we can't take the arrow γ, since it is pointing in the wrong direction.

Exp 2.2 ① The constant path $(i||i)$ at vertex i is the path of length $\ell = 0$ which never leaves vertex i. We denote it by e_i.

② An arrow $i \xrightarrow{\alpha} j$ is a path $(i|\alpha|j)$ of length 1. If $i=j$ then $i \circlearrowleft \alpha$ is called a loop.

③ A path of the form

$$\begin{tikzcd} i \arrow[r, "\alpha_1"] & \bullet \arrow[r, "\alpha_2"] & \bullet \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{\ell-1}"] & \bullet \arrow[llll, bend left, "\alpha_\ell"] \end{tikzcd}$$

given by $(i|\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_{\ell-1},\alpha_\ell|i)$ is called an oriented cycle. Thus a loop is an oriented cycle of length one.

2.1 Simple, Projective and Injective Representations

Let Q be a quiver without oriented cycles.

Def 2.2 Let i be a vertex of Q. Define representations $S(i)$, $P(i)$ and $I(i)$ as follows:

ⓐ $S(i)$ is of dimension one at vertex i, and zero at every other vertex; thus

$$S(i) = (S(i)_j, \varphi_\alpha)_{j\in Q_0, \alpha\in Q_1} \quad \text{where} \quad S(i)_j = \begin{cases} k & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \varphi_\alpha = 0 \text{ for all arrows } \alpha.$$

$S(i)$ is called the simple representation at vertex i.

ⓑ
$$P(i) = (P(i)_j, \varphi_\alpha)_{j\in Q_0, \alpha\in Q_1}$$

where $P(i)_j$ is the k-vector space with basis the set of all paths from i to j in Q; so the elements of $P(i)_j$ are of the form $\sum_c \lambda_c c$, where c runs over all paths from i to j, and $\lambda_c \in k$; and if $j \xrightarrow{\alpha} \ell$ is an arrow in Q, then $\varphi_\alpha : P(i)_j \to P(i)_\ell$ is the linear map defined on the basis by composing the paths from i to j with the arrow $j \xrightarrow{\alpha} \ell$. More precisely, the arrow α induces an injective map between the bases

$$\begin{aligned} \text{basis of } P(i)_j &\longrightarrow \text{basis of } P(i)_\ell \\ c = (i|\beta_1,\beta_2,\dots,\beta_s|j) &\longmapsto c\alpha = (i|\beta_1,\beta_2,\dots,\beta_s,\alpha|\ell) \end{aligned}$$

and φ_α is defined by

$$\varphi_\alpha\Big(\sum_c \lambda_c c\Big) = \sum_c \lambda_c c\alpha$$

$P(i)$ is called the projective representation at vertex i.

ⓒ
$$I(i) = (I(i)_j, \varphi_\alpha)_{j\in Q_0, \alpha\in Q_1}$$

where $I(i)_j$ is the k-vector space with basis the set of all paths from j to i in Q, so the elements of $I(i)_j$ are of the form $\sum_c \lambda_c c$, where c runs over all paths from j to i and $\lambda_c \in k$; and if $j \xrightarrow{\alpha} \ell$ is an arrow in Q, then $\varphi_\alpha : I(i)_j \to I(i)_\ell$ is the linear map defined on the basis by deleting the arrow $j \xrightarrow{\alpha} \ell$ from those paths from j to i which start with α and sending to zero the paths that do not start with α. More precisely, the arrow α induces a surjective map f between the bases

$$\begin{aligned} \text{basis of } I(i)_j &\xrightarrow{f} \text{basis of } I(i)_\ell \\ c = (j|\beta_1,\beta_2,\dots,\beta_s|i) &\longmapsto \begin{cases} (\ell|\beta_2,\dots,\beta_s|i) & \text{if } \beta_1 = \alpha \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and φ_α is defined by

$$\varphi_\alpha\Big(\sum_c \lambda_c c\Big) = \sum_c \lambda_c f(c)$$

$I(i)$ is called the injective representation at vertex i.

Note We need the hypothesis that Q has no oriented cycles, otherwise there would be a vertex i s.t. $P(i)$ is infinite-dimensional and thus not a representation in rep Q.
Exp: Q the quiver $1 \rightleftarrows 2$ then $P(1)$ and $P(2)$ would be infinite-dimensional.

Rk 2.1 Let $P(i) = (P(i)_j, \varphi_\alpha)$ be the projective representations at vertex i and let c be a path starting at i, say

$$c = (i \mid \beta_1, \beta_2, -, \beta_\ell \mid j)$$

Then we can define the map

$$\varphi_c : P(i)_i \to P(i)_j \qquad \varphi_c = \varphi_{\beta_\ell} \cdots \varphi_{\beta_2}\varphi_{\beta_1}$$

as the composition of the maps in the representation $P(i)$ along the path c. Then, if e_i denotes the constant path at vertex i, it follows from the definition of $P(i)$ that $\varphi_c(e_i) = c$

Rk 2.2 ① The projective representation at vertex i is the simple representation at vertex i iff there is no arrow α in Q s.t. $s(\alpha) = i$. Such vertices are called sinks of the quiver Q. Thus

$$S(i) = P(i) \iff i \text{ is a sink in } Q$$

② The injective representation at vertex i is the simple representation at vertex i iff there is no arrow α in Q s.t. $t(\alpha) = i$. Such vertices are called sources of the quiver Q. Thus

$$S(i) = I(i) \iff i \text{ is a source in } Q$$

Exp 2.3 Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[d] & 4 \arrow[l] \\ & & 5 & \end{tikzcd}$$

Then

$$S(3) \cong \begin{tikzcd} 0 \arrow[r] & 0 & k \arrow[l] \arrow[d] & 0 \arrow[l] \\ & & 0 & \end{tikzcd} \qquad P(3) \cong \begin{tikzcd} 0 \arrow[r] & k & k \arrow[l, "1"'] \arrow[d, "1"'] & 0 \arrow[l] \\ & & k & \end{tikzcd}$$

$$I(3) \cong \begin{tikzcd} 0 \arrow[r] & 0 & k \arrow[l] \arrow[d] & k \arrow[l, "1"'] \\ & & 0 & \end{tikzcd}$$

Exp 2.4 Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[rr] \arrow[dr] & & 3 \arrow[r] & 4 \\ & 2 \arrow[ur] & & \end{tikzcd}$$

$$P(1) \cong \begin{tikzcd} k \arrow[rr, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"] \arrow[dr] & & k^2 \arrow[r, "{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}"] & k^2 \\ & k \arrow[ur, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] & & \end{tikzcd} \qquad I(4) \cong \begin{tikzcd} k^2 \arrow[rr, "{[0\ 1]}"] \arrow[dr, "{[1\ 0]}"'] & & k \arrow[r, "1"] & k \\ & k \arrow[ur, "1"'] & & \end{tikzcd}$$

In category theory, a projective object is an object P s.t. the Hom functor $\mathrm{Hom}(P, -)$ maps surjective morphisms to surjective morphisms. The following proposition shows that $P(i)$ satisfies this condition.

Prop 2.3 Let $g : M \to N$ be a surjective morphism between representations of Q, and let $P(i)$ be the projective representation at vertex i. Then the map

$$g_* : \mathrm{Hom}(P(i), M) \longrightarrow \mathrm{Hom}(P(i), N) \quad \text{is surjective.}$$

In other words, if $f : P(i) \to N$ is any morphism, then there exists a morphism $h : P(i) \to M$ s.t. the diagram

$$\begin{tikzcd} & P(i) \arrow[dl, "h"'] \arrow[d, "f"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

commutes $(f = g \circ h = g_*(h))$

Cor 2.4 If P is projective, then any exact sequence of the form

$$0 \to L \to M \xrightarrow{g} P \to 0 \quad \text{splits.}$$

Proof Use prop. 2.3. with $f = 1_P$ (identity morphism) to get the commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, equal] & \\ M \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

Thus $1 = g \circ h$ and g is a retraction ■

In category theory, an injective object is an object I s.t. the Hom functor $\mathrm{Hom}(-, I)$ maps injective morphisms to surjective morphisms. The following proposition shows that $I(i)$ satisfies this condition.

Prop 2.5 Let $g: L \hookrightarrow M$ be an injective morphism between representations of Q, and let $I(i)$ be the injective representation at vertex i. Then the map

$$g_* : \mathrm{Hom}(M, I(i)) \to \mathrm{Hom}(L, I(i)) \quad \text{is surjective.}$$

In other words, if $f: L \to I(i)$ is any morphism, then there exists a morphism $h: M \to I(i)$ s.t. the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "g"] \arrow[d, "f"'] & M \arrow[dl, "h"] \\ & I(i) & \end{tikzcd}$$

commutes $(f = h \circ g = g^*(h))$

Cor 2.6 If I is injective then any exact sequence of the form

$$0 \to I \xrightarrow{g} M \to N \to 0 \quad \text{splits.}$$

Proof Use prop. 2.5 with $f = 1_I$ (identity morphism) to get a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "g"] \arrow[d, equal] & M \arrow[dl, "h"] \\ & I & \end{tikzcd}$$

Thus $1_I = h \circ g$ and g is a section.

A simple object in category is a nonzero object S that has no proper subobjects. The representations $S(i)$ have this property.
The next proposition states that sums of projective objects are projective and summands of projective objects are projective.

Prop 2.7 (1) Let P and P' be representations of Q. Then: $P \oplus P'$ is projective iff P and P' are projective
(2) Let I and I' be representations of Q. Then: $I \oplus I'$ is injective iff I and I' are injective

Proof We only prove (1) as (2) can be shown similar
"$\Leftarrow$" Let $g: M \to N$ be surjective in rep Q and $f: P \to N$ any morphism in rep$Q$. We consider the following diagram

$$\begin{tikzcd} & P \oplus P' \arrow[dl, "\exists h"'] \arrow[d, shift right, "pr_1"'] & \\ & P \arrow[u, shift right, "i_1"'] \arrow[dl, "h'"] \arrow[d, "f"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

With pr_1: projection on first summand, i_1: canonical injection and such that $pr_1 \circ i_1 = Id_P$

From Prop 2.3 we know since $P \oplus P'$ is projective that there exist a map h $P \oplus P' \to M$ st $f = f \circ \underbrace{pr_1 \circ i_1}_{Id_P} = g \circ h \circ i_1$
We next define $h': P \to M$ as $h' = h \circ i_1$ and we get that $f = g \circ h'$. This shows that P is projective. One can show in a similar way that P' is projective

"$\Leftarrow$" Again, $g: M \to N$ surj. morphism in rep Q and f $P \oplus P' \to N$ any morphism in rep Q. Consider the following commutative diagram

$$\begin{tikzcd} & P \arrow[d, "i_1"] \arrow[dl, "\exists h_1"'] \\ & P \oplus P' \arrow[d, "f"] \arrow[dl, "h"'] \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

with i_1: canonical injection

Since P is projective, by Prop 2.3 there exists a morphism h_1 s.t. $f \circ i_1 = g \circ h_1$. By symmetry there also exists a morphism h_2 $P' \to M$ s.t. $f \circ i_2 = g \circ h_2$

Now let us define $h = (h_1, h_2) \cdot P \oplus P' \to M$ by $h(p+p') = h_1(p) + h_2(p')$. We then obtain

$$g \circ h(p+p') \overset{\text{Def.}}{=} g \circ h_1(p) + g \circ h_2(p) = f \circ i_1(p) + f \circ i_2(p') = f(p+p')$$

This shows that $P \oplus P'$ is projective and concludes our proof ∎

Proposition 2.7 implies that if we know the indecomposable projective, resp. injective, representations, then we know all projective, resp. injective, representations.

The next Proposition tells us, that the representations $P(i)$ and $I(i)$ are in fact indecomposable

Prop 2.8 The representations $S(i)$, $P(i)$ and $I(i)$ are indecomposable

Proof First a short recall A representation $M \in \text{rep}\, Q$ is called indecomposable if $M \neq 0$ and M cannot be written as a direct sum of two nonzero representations

We will prove the statement for $S(i)$ and $P(i)$. The proof for $I(i)$ is similar.

$S(i)$ As $S(i)$ is simple, meaning it has no proper subobjects, the result follows directly

$P(i)$ $P(i) \overset{\text{Def.}}{=} (P(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$ As Q has no oriented cycles, we have $(P_i)_i = k$. Suppose by contradiction, that $P(i) = M \oplus N$ (is decomposable)

for some M, N in rep Q and suppose further wlog $(P_i)_i = M_i$ and $N_i = 0$. Let ℓ be a vertex of Q s.t. $N_\ell \neq 0$. Then $P(i)_\ell$ has a basis consisting of paths from $i \to \ell$.

Let $c = (i|\beta_1, \dots, \beta_s|\ell)$ be such a path. Let $\varphi_c = \varphi_{\beta_s} \cdots \varphi_{\beta_1}$ denote the composition of the linear maps of $P(i)$ along the path c

As $P(i)$ is the direct sum of M and N, $\varphi_c : M_i \oplus 0 \to M_\ell \oplus N_\ell$ sends unique basis element e_i of M_i to an element $\varphi_c(e_i)$ of M_ℓ

By remark 2.1 we know that $\varphi_c(e_i) = c$ and hence every basis element c of $P(i)_\ell$ lies in M_ℓ, which leads us to a contradiction.

We conclude that $P(i)$ is indecomposable indeed. ∎

Prop 2.9. A representation of Q is simple iff it is isomorphic to $S(i)$, for some $i \in Q_0$

Proof It's clear that $S(i)$ are simple representations, so lets prove the other direction

Let $M = (M_i, \varphi_\alpha)$ be any representation of Q. We want to show that there exists a vertex i s.t. $S(i)$ is a subrepresentation of M.

[As a short recall: L is a subrepresentation of a representation M if there exists an injective morphism $i: L \hookrightarrow M$]

We have to choose the vertex i carefully. We do not want to have a nonzero map in the representation M that starts at vertex i

For example, if i is a sink in the quiver [no arrow α in Q s.t. $S(\alpha) = i$], we have what we want

But we also need the representation M to be nonzero at the vertex i, which leads us to pick i as follows:

Let $i \in Q_0$ such that $M_i \neq 0$ and $M_j = 0$, whenever there is an arrow $i \xrightarrow{\alpha} j$ in Q. Such a vertex exists since Q has no oriented cycles.

Now choose any injective, linear map $f_i : (S_i)_i \cong k \to M_i$ and extend it trivially to a morphism $f: S(i) \to M$ by letting $f_j = 0$ if $i \neq j$. Note that f is actually a morphism as

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & S(i)_i \arrow[r] \arrow[d, "f_i"] & 0 \arrow[d] \\ M_\ell \arrow[r, "\varphi_\alpha"] & M_i \arrow[r, "\varphi_\beta"] & 0 \end{tikzcd}$$

commutes for all arrows $\ell \xrightarrow{\alpha} i$ and $i \xrightarrow{\beta} j$ in Q

Since f is injective, $S(i)$ is a subrepresentation of M, and thus, either $M \cong S(i)$ or M is not simple. ∎

Rk 2.10 Note that Prop. 2.9 does not hold if the quiver has oriented cycles. For example, if Q is the quiver $\alpha \circlearrowright 1$, then for each $\lambda \in k$, there is a simple representation $f_\lambda \circlearrowright k$, where f_λ is given by multiplication by λ.

The vector space of a vertex i of any representation can be described as a space of morphisms using the projective representation $P(i)$ as shown in the next theorem.

Thm. 2.11 Let $M = (M_i, \varphi_\alpha)$ be any representation of Q. Then, for any vertex i in Q, there is an isomorphism of vector spaces:

$$\mathrm{Hom}(P(i), M) \cong M_i$$

Proof Let $e_i = (i\|i)$ be the constant path at i. Then $\{e_i\}$ is a basis of the vector space $P(i)_i$. Define a map $\Phi: \mathrm{Hom}(P(i), M) \to M_i$, $f = (f_j)_{j \in Q_0} \mapsto f_i(e_i)$.
If f is a morphism from $P(i)$ to M, then f_i is a linear map from $P(i)_i$ to M_i, which shows that Φ is well-defined, since $e_i \in P(i)_i$.
What we want to prove: Φ is an isomorphism of vector spaces
Let us use the notation $P(i) = (P(i)_j, \varphi_\alpha)$

(1) Φ is linear:
Let $f, g \in \mathrm{Hom}(P(i), M)$, two morphisms. Then $\Phi(f+g) \overset{\text{Def}}{=} (f+g)_i(e_i) = f_i(e_i) + g_i(e_i) = \Phi(f) + \Phi(g)$
Let $\lambda \in k$, then $\Phi(\lambda f) = (\lambda f)_i(e_i) = \lambda f_i(e_i) = \lambda \Phi(f)$

(2.) Φ is injective
If $0 = \Phi(f) = f_i(e_i)$, then the linear map f_i sends the basis $\{e_i\}$ to zero, and therefore f_i is the zero map. We will show that $f_j: P(i)_j \to M$ is the zero map for any vertex j, which will show that Φ is injective. By the definition of $P(i)$, the vector space $P(i)_j$ has a basis consisting of all paths from i to j.
Let $c = (i\,|\,\alpha_1, \ldots, \alpha_t\,|\,j)$ be such a basis element, and consider the maps $\varphi_c = \varphi_{\alpha_t} \circ \cdots \circ \varphi_{\alpha_1}$ and $\varphi'_c = \varphi'_{\alpha_t} \circ \cdots \circ \varphi'_{\alpha_1}$ defined as composition of the maps along the path c of the repr. $P(i)$ & M, resp.
From the definition of $P(i)$ it follows that $\varphi_c(e_i) = c$. Since f is a morphism of repr, $f_j \varphi_c = \varphi'_c f_i$ and as $f_i(e_i) = 0$ it follows that f_j maps c to zero.
Since c was chosen an arbitrary basis element of $P(i)_j$, it follows that $f_j = 0$

(3) Φ is surjective:
Let $m_i \in M_i$. We want to construct a morphism $f: P(i) \to M$ s.t. $f_i(e_i) = m_i$. Let us start, by fixing its component $f_i: P(i)_i \to M_i$ by requiring the desired condition $f_i(e_i) = m_i$.
Since $\{e_i\}$ is a basis of $P(i)_i$, this condition defines the linear map f_i in an unique way. We can extend f_i to a morphism $f = (f_j)_{j \in Q_0}$ by "following the paths in Q".
More precisely, for any path c from i to a vertex j in Q, put $f_j(c) = \varphi'_c(m_i)$. This defines each map f_j on a basis of $P(i)_j$, and we can extend this map linearly to the whole vector space $P(i)_j$. From our constructions it follows that f is a morphism of representations.
Thus, $f \in \mathrm{Hom}(P(i), M)$ and $\Phi(f) = m_i$, Φ is surjective ∎

Corollary 2.12 follows as an immediate consequence of the Theorem above, and shows us how we can describe the morphisms between projective repr.

Cor 2.12 Let i and j be vertices in Q.
(1) The vector space $\mathrm{Hom}(P(i), P(j))$ has a basis consisting of all paths from j to i in Q. In particular, $\mathrm{End}(P(i)) = \mathrm{Hom}(P(i), P(i)) \cong k$
(2) If $A = \oplus_{i \in Q_0} P(i)$, then the vector space $\mathrm{End}(A) = \mathrm{Hom}(A, A)$ has a basis consisting of all paths in Q

Proof From Thm. 2.11 we get that $\mathrm{Hom}(P(i), P(j))$ is isomorphic to $P(j)_i$, and this vector space has a basis consisting of all the paths from j to i in Q.
The fact that Q has no oriented cycles gives us that $\mathrm{End}(P(i))$ is of dimension 1. Therefore $\mathrm{End}(P(i)) \cong k$ and this proves (1) & (2) as direct consequences ∎

Cor 2.14 The representation $P(j)$ is a simple representation iff $\mathrm{Hom}(P(i), P(j)) = 0$ for all $i \neq j$.

Proof The representation $P(j)$ is simple iff j is a sink [see the definition of a sink again in Rk. 2.2]. This means that there are no paths from j to any other vertex i.
The statement then follows from Corollary 2.12 ∎

Problems

2.2 Compute the indecomposable projective representations $P(i)$ and the indecomposable injective representations $I(i)$ for the following quivers:

From proposition 2.8 we know that the representations $S(i)$ and $P(i)$ for all $i \in Q_0$ are indecomposable.

①

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[dr] & 4 & 5 \arrow[l] \\ & 6 \arrow[ur] & & 7 & \end{tikzcd}$$

$$P(1) \cong \begin{tikzcd} k \arrow[r, "1"] & k \arrow[r, "1"] & k \arrow[r, "1"] \arrow[dr, "1"] & k & 0 \arrow[l] \\ & 0 \arrow[ur] & & k & \end{tikzcd} \qquad I(1) \cong \begin{tikzcd} k \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[dr] & 0 & 0 \arrow[l] \\ & 0 \arrow[ur] & & 0 & \end{tikzcd}$$

$$P(2) \cong \begin{tikzcd} 0 \arrow[r] & k \arrow[r, "1"] & k \arrow[r, "1"] \arrow[dr, "1"] & k & 0 \arrow[l] \\ & 0 \arrow[ur] & & k & \end{tikzcd} \qquad I(2) \cong \begin{tikzcd} k \arrow[r, "1"] & k \arrow[r] & 0 \arrow[r] \arrow[dr] & 0 & 0 \arrow[l] \\ & 0 \arrow[ur] & & 0 & \end{tikzcd}$$

$$P(3) \cong \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & k \arrow[r, "1"] \arrow[dr, "1"] & k & 0 \arrow[l] \\ & 0 \arrow[ur] & & k & \end{tikzcd} \qquad I(3) \cong \begin{tikzcd} k \arrow[r, "1"] & k \arrow[r, "1"] & k \arrow[r] \arrow[dr] & 0 & 0 \arrow[l] \\ & k \arrow[ur, "1"] & & 0 & \end{tikzcd}$$

$$P(4) \cong \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[dr] & k & 0 \arrow[l] \\ & 0 \arrow[ur] & & 0 & \end{tikzcd} \qquad I(4) \cong \begin{tikzcd} k \arrow[r, "1"] & k \arrow[r, "1"] & k \arrow[r, "1"] \arrow[dr] & k & k \arrow[l, "1"'] \\ & k \arrow[ur, "1"] & & 0 & \end{tikzcd}$$

$$P(5) \cong \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[dr] & k & k \arrow[l, "1"'] \\ & 0 \arrow[ur] & & & \end{tikzcd} \qquad I(5) \cong \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[dr] & 0 & k \arrow[l] \\ & 0 \arrow[ur] & & 0 & \end{tikzcd}$$

$$P(6) \cong \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & k \arrow[r, "1"] \arrow[dr, "1"] & k & 0 \arrow[l] \\ & k \arrow[ur, "1"] & & k & \end{tikzcd} \qquad I(6) \cong \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[dr] & 0 & 0 \arrow[l] \\ & k \arrow[ur] & & 0 & \end{tikzcd}$$

$$P(7) \cong \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[dr] & 0 & 0 \arrow[l] \\ & 0 \arrow[ur] & & k & \end{tikzcd} \qquad I(7) \cong \begin{tikzcd} k \arrow[r, "1"] & k \arrow[r, "1"] & k \arrow[r] \arrow[dr, "1"] & 0 & 0 \arrow[l] \\ & k \arrow[ur, "1"] & & k & \end{tikzcd}$$

② $1 \leftleftarrows 2 \leftleftarrows 3$

$$P(1) \cong k \leftleftarrows 0 \leftleftarrows 0 \qquad I(1) \cong \begin{tikzcd} k & k^2 \arrow[l, shift right, "{[0\ 1]}"'] \arrow[l, shift left, "{[1\ 0]}"'] & k^4 \arrow[l, shift right, "{\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}}"'] \arrow[l, shift left, "{\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}}"'] \end{tikzcd}$$

$$P(2) \cong \begin{tikzcd} k^2 & k \arrow[l, shift right, "{\begin{bmatrix}1\\0\end{bmatrix}}"'] \arrow[l, shift left, "{\begin{bmatrix}0\\1\end{bmatrix}}"'] & 0 \arrow[l, shift right] \arrow[l, shift left] \end{tikzcd} \qquad I(2) \cong \begin{tikzcd} 0 & k \arrow[l, shift right] \arrow[l, shift left] & k^2 \arrow[l, shift right, "{[1\ 0]}"'] \arrow[l, shift left, "{[0\ 1]}"'] \end{tikzcd}$$

$$P(3) \cong \begin{tikzcd} k^4 & k^2 \arrow[l, shift right, "{\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}}"'] \arrow[l, shift left, "{\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}}"'] & k \arrow[l, shift right, "{\begin{bmatrix}1\\0\end{bmatrix}}"'] \arrow[l, shift left, "{\begin{bmatrix}0\\1\end{bmatrix}}"'] \end{tikzcd} \qquad I(3) \cong 0 \leftleftarrows 0 \leftleftarrows k$$

③

$$\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] \arrow[d] & 3 \arrow[dl] \\ & 4 & \end{tikzcd}$$

$$P(1) \cong \begin{tikzcd} k \arrow[r, "1"] \arrow[rr, bend left, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"] & k \arrow[r, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}"'] & k^2 \arrow[dl, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}"] \\ & k^3 & \end{tikzcd}$$

$$I(1) \cong \begin{tikzcd} k \arrow[r] \arrow[rr, bend left] & 0 \arrow[r] \arrow[d] & 0 \arrow[dl] \\ & 0 & \end{tikzcd}$$

$$P(2) \cong \begin{tikzcd} 0 \arrow[r] \arrow[rr, bend left] & k \arrow[r, "1"] \arrow[d, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"'] & k \arrow[dl, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] \\ & k^2 & \end{tikzcd}$$

$$I(2) \cong \begin{tikzcd} k \arrow[r, "1"] \arrow[rr, bend left] & k \arrow[r] \arrow[d] & 0 \arrow[dl] \\ & 0 & \end{tikzcd}$$

$$P(3) \cong \begin{tikzcd} 0 \arrow[r] \arrow[rr, bend left] & 0 \arrow[r] \arrow[d] & k \arrow[dl, "1"] \\ & k & \end{tikzcd}$$

$$I(3) \cong \begin{tikzcd} k^2 \arrow[r, "\begin{bmatrix} 0 & 1 \end{bmatrix}"'] \arrow[rr, bend left, "\begin{bmatrix} 1 & 0 \end{bmatrix}"] & k \arrow[r, "1"] \arrow[d] & k \arrow[dl] \\ & 0 & \end{tikzcd}$$

$$P(4) \cong \begin{tikzcd} 0 \arrow[r] \arrow[rr, bend left] & 0 \arrow[r] \arrow[d] & 0 \arrow[dl] \\ & k & \end{tikzcd}$$

$$I(4) \cong \begin{tikzcd} k^3 \arrow[r, "\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}"'] \arrow[rr, bend left, "\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}"] & k^2 \arrow[r, "\begin{bmatrix} 1 & 0 \end{bmatrix}"'] \arrow[d, "\begin{bmatrix} 0 & 1 \end{bmatrix}"] & k \arrow[dl, "1"] \\ & k & \end{tikzcd}$$