Characters I

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G is always assumed to be finite!

<u>Remark</u>: $\varphi: G \to GL_n(\mathbb{C})$ a rep., then $\varphi_g = (\varphi_{ij}(g))_{i,j}$ with $\varphi_{ij}(g) \in \mathbb{C}$. To φ belong therefore n^2 functions $\varphi_{ij}: G \to \mathbb{C}$.

<u>Def.</u> $L(G) := \{ f : f: G \to \mathbb{C} \}$

<u>Remark</u>: $L(G)$ is an inner product space with

$$(f_1 + f_2)(g) = f_1(g) + f_2(g)$$

$$(cf)(g) = c \cdot f(g)$$

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \cdot \overline{f_2(g)}$$

$L(G)$ is called the group algebra of G

<u>Prop.</u> $\varphi: G \to GL(V)$, $\rho: G \to GL(W)$ two rep. $T: V \to W$ linear. Then

a) $T^\# := \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_g \in \mathrm{Hom}_G(\varphi, \rho)$

b) $T \in \mathrm{Hom}_G(\varphi, \rho) \Rightarrow T^\# = T$

c) $P: \mathrm{Hom}(V, W) \to \mathrm{Hom}_G(\varphi, \rho)$ with $P(T) = T^\#$ is surjective and linear.

Proof: a)

$$T^{\#}\varphi_h = \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T \varphi_g \varphi_h = \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T \varphi_{gh}$$

Set $x = gh$ (as g varies over G, so does x)

$$\Rightarrow g^{-1} = h\cdot x^{-1} \Rightarrow T^{\#}\varphi_h = \left(\sum_{x\in G} \rho_{hx^{-1}} T \varphi_x\right)\frac{1}{|G|}$$

$$= \frac{1}{|G|}\sum_{x\in G} \rho_h \rho_{x^{-1}} T \varphi_x = \rho_h \cdot \frac{1}{|G|}\sum_{x\in G} \rho_{x^{-1}} T \varphi_x$$

$$= \rho_h T^{\#} \Rightarrow T^{\#} \in \mathrm{Hom}(\varphi, \rho)$$

b) $T \in \mathrm{Hom}_G(\varphi, \rho)$

$$T^{\#} = \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T \varphi_g = \frac{1}{|G|}\sum_{g\in G} \overbrace{\rho_{g^{-1}}\rho_g}^{=I} T$$

$$= \frac{1}{|G|}\sum_{g\in G} T = \frac{1}{|G|}\cdot |G|\cdot T = T$$

c) P is clearly linear. Take $T \in \mathrm{Hom}_G(\varphi, \rho)$

$\Rightarrow T^{\#} = P(T) = T \Rightarrow P$ is surjective. $\blacksquare$

Prop. $\varphi: G \to GL(V)$, $\rho: G \to GL(W)$ irreducible rep, $T: V \to W$ linear.

a) $\varphi \not\sim \rho$, then $T^{\#} = 0$

b) $\varphi = \rho$, then $T^{\#} = \dfrac{\mathrm{Tr}(T)}{\deg(\varphi)}\cdot I$

Proof a) $\varphi \not\sim \rho$. With Schur's lemma:

$\mathrm{Hom}_G(\varphi, \rho) = 0 \Rightarrow T^{\#} = 0$

b) $\varphi = \rho$, with Schur's lemma: $T^{\#} = \lambda\cdot I$ $(\lambda \in \mathbb{C})$

$T^{\#}: V \to V$, $\mathrm{Tr}(\lambda I) = \lambda\cdot \mathrm{Tr}(I) = \lambda\cdot \dim(V)$

$$= \lambda \cdot \dim(\varphi) \Rightarrow T^{\#} = \underbrace{\frac{\mathrm{Tr}(T^{\#})}{\dim(\varphi)}}_{=\lambda}\cdot I$$

We have $Tr(A\cdot B) = Tr(B\cdot A)$

$$Tr(T^\#) = \frac{1}{|G|}\sum_{g\in G} Tr(\varphi_{g^{-1}} T \varphi_g)$$

$$= \frac{1}{|G|}\sum_{g\in G} Tr(\varphi_{g^{-1}}\varphi_g T) = \frac{1}{|G|}\sum_{g\in G} Tr(T) = Tr(T)$$

$$\Rightarrow T^\# = \frac{Tr(T)}{\dim(\varphi)} \cdot I$$

$\blacksquare$

<u>Notation</u>: $E_{ij} \in M_{mn}(\mathbb{C})$ matrix with ij-entry $= 1$, 0 else.

<u>Lemma</u>: $A \in M_{rm}(\mathbb{C})$, $B \in M_{ns}(\mathbb{C})$, $E_{ki} \in M_{mn}(\mathbb{C})$

Then $(A\cdot E_{ki} B)_{\ell j} = a_{\ell k} b_{ij}$

<u>Proof</u>: $(A\cdot E_{ki} B)_{\ell j} = \sum_{xy} a_{\ell x} \underbrace{(E_{ki})_{xy}}_{=0 \text{ if } x\neq k \text{ and } y\neq i} b_{yj}$

$= a_{\ell k} b_{ij}$

<u>Notation</u>: $U_n(\mathbb{C})$ is the group of all $n\times n$ unitary matrices ($UU^\# = I$)

<u>Lemma</u>: $\varphi: G \to U_n(\mathbb{C})$, $\rho: G \to U_m(\mathbb{C})$ unitary rep. $A = E_{ki} \in M_{mn}(\mathbb{C})$, then

$$A^\#_{\ell j} = \langle \varphi_{ij}, \rho_{k\ell}\rangle$$

<u>Proof</u> $\rho_{g^{-1}} = \rho_g^{-1} = \rho_g^\# \Rightarrow \rho_{\ell k}(g^{-1}) = \overline{\rho_{k\ell}(g)}$

$$A^\#_{\ell j} = \frac{1}{|G|}\sum_{g\in G} (\rho_{g^{-1}} E_{ki} \varphi_g)_{\ell j} = \frac{1}{|G|}\sum_{g\in G} (\rho_{g^{-1}})_{\ell k} (\varphi_g)_{ij} = \frac{1}{|G|}\sum_{g\in G} \overline{\rho_{k\ell}(g)}\, \varphi_{ij}(g) = \langle \varphi_{ij}, \rho_{k\ell}\rangle$$

Theorem (Schur orthogonality relations) $\varphi : G \to U_n(\mathbb{C})$

$\rho : G \to U_m(\mathbb{C})$ inequivalent, irreducible rep.

a) $\langle \varphi_{ij}, \rho_{kl} \rangle = 0$

b) $\langle \varphi_{ij}, \varphi_{kl} \rangle = \begin{cases} 1/n & \text{if } i=k, j=l \\ 0 & \text{else} \end{cases}$.

Proof:

a) $A = E_{ki} \in M_{mn}(\mathbb{C})$, then $A^{\#} = 0$

(since the two rep. are inequivalent), $A^{\#}_{e_j} = \langle \varphi_{ij}, \rho_{kl} \rangle$

according to lemma. $\Rightarrow \langle \varphi_{ij}, \rho_{kl} \rangle = 0$

b) take $\varphi = \rho$, $A^{\#} = \dfrac{\mathrm{Tr}(E_{ki})}{\dim(\varphi)} \cdot I = \dfrac{\mathrm{Tr}(E_{ki})}{n} I$

with Prop. from above. Once again with

lemma $A^{\#}_{e_j} = \langle \varphi_{ij}, \rho_{kl} \rangle = \langle \varphi_{ij}, \varphi_{kl} \rangle$

Case 1: $j \neq l$ then $I_{e_j} = 0 \Rightarrow 0 = A^{\#}_{e_j}$

$= \langle \varphi_{ij}, \varphi_{kl} \rangle$.

Case 2: $i \neq k$, E_{ki} has only zeros on

the diagonal $\Rightarrow$ ~~trace (E~~ $\mathrm{Tr}(E_{ki}) = 0$

$\Rightarrow 0 = A^{\#}_{e_j} = \langle \varphi_{ij}, \varphi_{kl} \rangle$

Case 3: $j = l$, $i = k$ $\mathrm{Tr}(E_{ki}) = 1$

$\Rightarrow A^{\#} = \frac{1}{n} \cdot I \Rightarrow A^{\#}_{e_j} = \langle \varphi_{ij}, \varphi_{kl} \rangle$

$= \frac{1}{n}$ $\square$

Corollary: φ irreducible, unitary rep. of G of degree d. Then, the d^2 functions $\{\sqrt{d}\,\varphi_{ij} \cdot 1 \leq i,j \leq n\}$ form an orthonormal set

Prop. $\varphi^{(1)}, \ldots, \varphi^{(s)}$ a complete set of representatives of the equivalence classes of irreducible rep. of G; $d_i = \deg \varphi^{(i)}$. Then, the functions

$$\{\sqrt{d_k}\,\varphi_{ij}^{(k)} \mid 1 \leq k \leq s,\ 1 \leq i,j \leq d_k\}$$

form an orthonormal set in $L(G)$ and

$$s = d_1^2 + \ldots + d_s^2 \leq |G|$$

Proof. Every equivalence class contains a unitary rep., $\dim L(G) = |G| \Rightarrow$ every linearly independent set of vectors in $L(G)$ contains at most $|G|$ elements. With theorem:

The $d_1^2 + \ldots + d_s^2$ functions

$$\{\sqrt{d_k}\,\varphi_{ij}^{(k)} \mid 1 \leq k \leq s,\ 1 \leq i,j \leq d_k\}$$

form an orthonormal set in $L(G)$.

$$\Rightarrow \quad d_1^2 + \ldots + d_s^2 \leq |G|$$

Moreover $d_i \geq 1 \quad \forall i$

$$\Rightarrow \quad s \leq d_1^2 + \ldots + d_s^2$$