## EXERCISES 5: LECTURE FOUNDATIONS OF MATHEMATICS

**Exercise 1.** Let $X$ be a set, and let $\mathfrak{P}(X)$ denote its power set. Show:

(a) $(\mathfrak{P}(X), \subset)$ is an ordered set.

(b) $(\mathfrak{P}(X), \subset)$ is a totally ordered set if and only if $X = \emptyset$ or $X = \{a\}$.

**Exercise 2.** Let $X$ be a set, and let $\mathfrak{A} \neq \emptyset$ be a subset of $\mathfrak{P}(X)$. Show that $\sup(\mathfrak{A}) = \bigcup \mathfrak{A}$ and $\inf(\mathfrak{A}) = \bigcap \mathfrak{A}$.

**Exercise 3.** Let $(\mathbb{N}_0^1, 0_1, \nu_1)$ and $(\mathbb{N}_0^2, 0_2, \nu_2)$ denote two sets for which the Peano axioms hold. That is, for $i \in \{1, 2\}$, the set $\mathbb{N}_0^i$ has a fixed element $0_i$ and an injective map $\nu_i \colon \mathbb{N}_0^i \to \mathbb{N}_0^i \setminus \{0_i\}$ such that

$$(\star)\colon\ ((0_i \in N_i \subset \mathbb{N}_i) \wedge (n_i \in N_i \Rightarrow \nu_i(n_i) \in N_i)) \Rightarrow N_i = \mathbb{N}_i$$

hold. Show that there exists a bijection $\phi \colon \mathbb{N}_0^1 \to \mathbb{N}_0^2$ with $\phi(0_1) = 0_2$ and $\nu_2 \circ \phi = \phi \circ \nu_1$. (That is, $\mathbb{N}_0$ is unique up to isomorphism.)

**Exercise 4.** Let $(\mathbb{N}_0^1, 0_1, \nu_1)$ and $(\mathbb{N}_0^2, 0_2, \nu_2)$ be as in Exercise 3, but $(\star)$ does not need to hold. Is there still a bijection as in Exercise 3? Justify your answer.

**Submission of the exercise sheet**: 21.Oct.2019 before the lecture. **Return of the exercise sheet**: 24.Oct.2019 during the exercise sessions.