Lecture 7 Additive, linear and abelian cat II – adding $\otimes$

Recall: Last time we have seen "categorification" of abelian groups, either using $\oplus$ only or additionally with homological algebra

Today: "categorifications" of rings/algebras

Def 7.1 A locally finite, $\mathbb{K}$-linear, abelian, rigid category $\mathcal{C}$ with bilinear $\otimes: \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is called a multi tensor cat

Fact: This implies that $\otimes$ is biexact

If $\mathrm{End}_{\mathcal{C}}(\mathbb{1})\cong\mathbb{K}$, then $\mathcal{C}$ is called a tensor cat.

$\oplus$ rather than abelian version of this are called fiat. Of course tensor $\Rightarrow$ fiat, but $\not\Leftarrow$

Examples — This is a vast generalization of G-fdmod, fd reps of a finite group G, which is the first example (for first G-fdproj would be ok)

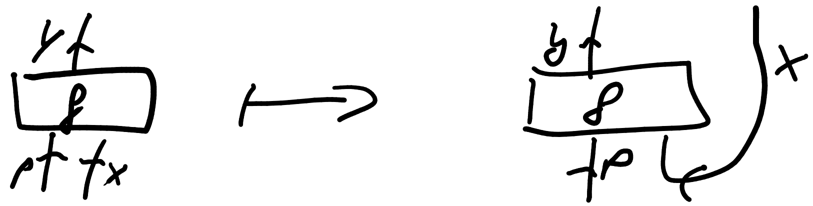
- fd$\mathrm{Vect}_{\mathbb{K}}$
- $\mathrm{Vect}^{\omega}(G)$ (or rather its matrix version - I stop making this distinction)
- H-fdmod for some fd Hopf algebra (H-fdproj first)

Beware: Since having rigidity usually requires fd things, I will make this the default and drop the fd, e.g. I write $\mathrm{Vect}_{\mathbb{K}}$ not fd$\mathrm{Vect}_{\mathbb{K}}$

Lemma 7.2 If $P \in \mathrm{Proj}(\mathcal{C})$ and $X \in \mathcal{C}$, then PX and XP are projective.

(multi) tensor or fiat from now on

Proof $\mathrm{Hom}_{\mathcal{C}}(PX, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(P, Y \otimes {}^{\vee}X)$



The easiest tensor categories are those where simple $\Leftrightarrow$ inde. These are called semisimple

Theorem 7.3 (Maschke) $\mathcal{C}$ is semisimple $\Leftrightarrow \mathbb{1} \in \mathrm{Proj}(\mathcal{C})$

Proof The above lemma implies that, if $\mathbb{1}$ is projective, then everything is.

Remark This is the analog of Maschke's theorem for finite groups: $\mathbb{K}[G]$ is semisimple $\Leftrightarrow$ $\underline{\mathbb{1}}$ is projective $\Leftrightarrow$ $\mathrm{char}(\mathbb{K}) \nmid |G|$

↑ triv rep

In fact, his proof (dates back > 100 years) is "the same": He constructed an idempotent splitting off the trivial rep from the regular rep which implies that $\mathbb{1}$ is projective.

[Note here: $P = P_1 \oplus P_2$ implies P_1 and P_2 to be projective since $\mathrm{Hom}_{\mathcal{C}}(P, \_) \cong \mathrm{Hom}_{\mathcal{C}}(P_1, \_) \oplus \mathrm{Hom}_{\mathcal{C}}(P_2, \_)$

proj

Proposition 7.4 $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is semisimple

Proof By $\textcircled{f}\,{}^{\textcircled{g}} : {}^{\textcircled{f}}\,\textcircled{g}$, we already know that $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is commutative. Thus, it remains to show that $f \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ with $f^2 = 0$ implies $f = 0$.

Let $J = \mathrm{im}(f)$ and $K = \ker(f)$.

This uses the easy $\mathrm{im}(fg) \cong \mathrm{im}(f)\,\mathrm{im}(g)$

We first observe $JJ \cong \mathrm{im}(ff) \cong \mathrm{im}(0) \cong 0$.

Now $KJ \,(\cong JK) \cong 0$ because it is both, the image of $\mathrm{id}\, f$ on $K\mathbb{1}$ as well as of $f\,\mathrm{id}$. Thus, multiplying the exact sequence

$$K \to \mathbb{1} \longrightarrow J$$

with J shows $J = 0$

Question: Can one classify tensor cats with e.g. a fixed number of simples?

Sadly this is much harder than classifying simple finite groups (so, very hard).

Example / Remark

Let $\mathbb{K}=\mathbb{C}$. Then one can actually show that $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{1},\mathbb{1})=0$ which implies

"For $\mathbb{K}=\mathbb{C}$, $\mathrm{Vect}_{\mathbb{C}}$ is the only tensor cat with one simple object"

As we have seen $\mathbb{Z}/p\mathbb{Z}$-mod over $\mathbb{F}_p$ is also of such type, so even this is wrong in char >0

For now let $\mathcal{C}$ be an $\oplus$-cat. $e^2 = e$ (idempotent) makes sense and $(\mathrm{id}-e)^2 = \mathrm{id}-e$ is also an idempotent.

Question: Can we make sense of the formula $X \simeq \mathrm{im}(e) \oplus \mathrm{im}(1-e)$?

In general, no since $\mathrm{im}(e)$ does not make sense. However:

$\simeq$ $X \simeq \mathrm{im}(e) \oplus \ker(e)$

Def 7.5 The idempotent completion $\mathrm{Kar}(\mathcal{C})$ of $\mathcal{C}$ is the $\oplus$-cat defined as follows

- Objects $(X, e : X \to X)$, e idempotent
- Mor. $f \in \mathcal{C}$ such that

$$\begin{tikzcd}
(X,e) \arrow[r, "f"] \arrow[d, "e"'] & (X',e') \arrow[d, "e'"] \\
(X,e) \arrow[r, "f"'] & (X',e')
\end{tikzcd}$$

- $\mathrm{id}_{(X,e)} = e$

An object of $\mathrm{Kar}(\mathcal{C})$ can also be written as $\mathrm{im}\, e$ since it is the cat. theoretical image of e

Remark $\mathrm{Kar}(\mathcal{C})$ is the smallest extension of $\mathcal{C}$ (note the full embedding $\mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C})$, $X \mapsto (X, id_X)$) were images of idempotents exist.

This is still much weaker than being abelian (eg no kernels or cokernels), and $\mathrm{Kar}(\mathcal{C})$ is rigid, pivotal etc. if $\mathcal{C}$ is

Lemma 7.6 Let $e: X \to X$ be an idempotent in $\mathcal{C}$.

Then $X \cong \mathrm{im}(e) \oplus \mathrm{im}(1-e)$ in $\mathrm{Kar}(\mathcal{C})$.

Proof $X \rightleftarrows \mathrm{im}(e) \oplus \mathrm{im}(1-e) \longrightarrow \begin{pmatrix} e & 1-e \end{pmatrix} \leftarrow \begin{pmatrix} e \\ 1-e \end{pmatrix}$

The following says we can pass to $\mathrm{Kar}(-)$ without too much cost.

Proposition 7.7 For any $F : \mathcal{C} \to \mathcal{D}$ $\exists!$ (up to iso) $\mathrm{Kar}(F) : \mathrm{Kar}(\mathcal{C}) \to \mathrm{Kar}(\mathcal{D})$ such that

$$\begin{tikzcd}
\mathrm{Kar}(\mathcal{C}) \arrow[r, "\mathrm{Kar}(F)"] & \mathrm{Kar}(\mathcal{D}) \\
\mathcal{C} \arrow[u, hook] \arrow[r, "F"'] & \mathcal{D} \arrow[u, hook]
\end{tikzcd}$$

commutes and such that "$\mathrm{Kar}\,F$ has the same properties as F"

(eg if F is an $\otimes$-equi; then so is $\mathrm{Kar}(F)$)

Proof $\mathrm{Kar}(F)$ is defined via

$$\big((X, e) \xrightarrow{f} (X', e') \big) \longmapsto (FX, Fe) \xrightarrow{Ff} (FX', Fe')$$

Examples - $\mathrm{Kar}(\mathcal{C}) \simeq \mathrm{Kar}(\mathrm{Kar}(\mathcal{C}))$

(such cats are called idempotent complete)

- $K(\mathcal{C}) \simeq \mathcal{C}$ if $\mathcal{C}$ is abelian

- Let $A = \mathbb{K}[G]$. Let $\langle A \rangle_{\oplus,\otimes} \subset A\text{-mod}$ be the full subcat generated by the regular rep A. Then, in general,

$$\langle A \rangle_{\oplus,\otimes} \underset{\neq}{\hookrightarrow} \mathrm{Kar}\langle A \rangle_{\oplus,\otimes} \simeq A\text{-proj} \underset{\neq}{\hookrightarrow} A\text{-mod}$$

Note that we also have X is inde in $\mathrm{Kar}(\mathcal{C})$ $\Leftrightarrow$ $\mathrm{End}_{\mathcal{C}}(X)$ has no non-trivial idempotents

(since otherwise $X \simeq \mathrm{im}(e) \oplus \mathrm{im}(1-e)$)

Here is some "numerical" data associated to $\mathcal{C}$:

Def 7.8 The Grothendieck groups $K_0(\mathcal{C})$ and $G_0(\mathcal{C})$ are defined to be the $\mathbb{Z}$-modules obtained as follows.

Assume that $\mathcal{C}$ is Krull-Schmidt and $\oplus$, then

$$K_0(\mathcal{C}) := \mathbb{Z}\{[M]\} / [X \oplus Y] = [X] \oplus [Y]$$

Assume that $\mathcal{C}$ is locally finite, abelian, then

$$G_0(\mathcal{C}) = \mathbb{Z}\{[M]\} / [X] = [Y] \oplus [Z] \text{ where}$$

$Y \hookrightarrow X \twoheadrightarrow Z$ is exact

Lemma 7.9 If $\mathcal{C}$ is additionally a $\otimes$-cat, then $K_0(\mathcal{C})$ and $G_0(\mathcal{C})$ are both rings rather than $\mathbb{Z}$-modules

Proof Define $[X]\cdot[Y] := [X \otimes Y]$

Note The above is a justification why we can think of $\oplus$ or abelian cats as "cate-categorification" of $\mathbb{Z}$-modules / abelian groups and of first or tensor categories as "categorification" of rings / algebras

We can also categorify homomorphisms:

Lemma 7.10 Any $\oplus, \otimes$ functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ gives rise to a ring hom $[\mathcal{F}]: K_0(\mathcal{C}) \to K_0(\mathcal{D})$.
Similarly, any exact $\otimes$ functor gives a ring hom $[\mathcal{F}]: G_0(\mathcal{C}) \to G_0(\mathcal{D})$

Proof: Because of $\mathcal{F}(X \oplus Y) \simeq \mathcal{F}(X) \oplus \mathcal{F}(Y)$ and $\mathcal{F}(XY) \simeq \mathcal{F}(X)\mathcal{F}(Y)$

Remark All of the various "finiteness condition" are to avoid the Eilenberg swindle: If $\mathcal{F} = B \oplus A \oplus B \oplus A \oplus \dots$, the $A \oplus \mathcal{F} \simeq \mathcal{F}$, so $[A] = 0$ in K_0
This can be seen as a cat. version of

$$1 = 1 + (-1 + 1) + (-1 + 1) + \dots = (1 - 1) + (1 - 1) + \dots = 0$$

Lemma 7.10 $K_0(\mathcal{C})$ is a free $\mathbb{Z}$-module with basis $\{[I] \mid I \text{ indec}\}$.
$G_0(\mathcal{C})$ is a free $\mathbb{Z}$-module with basis $\{[S] \mid S \text{ simple}\}$

Proof: Clear.

Example If $\mathcal{C} = \mathrm{Vect}_{\mathbb{K}}$, then $K_0(\mathcal{C}) \simeq G_0(\mathcal{C}) \simeq \mathbb{Z}$

The iso take $[\mathbb{K}] \longmapsto 1 \in \mathbb{Z}$

This is a map of rings since

$[\mathbb{K}^m] = [\mathbb{K} \oplus \ldots \oplus \mathbb{K}] = m[\mathbb{K}]$, $[\mathbb{K}^n \oplus \mathbb{K}^n] = [\mathbb{K}^n] + [\mathbb{K}^n]$

and $[\mathbb{K}^m \otimes \mathbb{K}^n] = [\mathbb{K}^m][\mathbb{K}^n]$

Let $[X:S_i]$ and $(X:I_j)$ denote the multiplicities of simple (in a JH filt) respecting inde. in an object X

Then, by definition, $[X] = \sum [X:S_i][S_i]$ in G_0

$$[X] = \sum (X:I_j)[I_j] \text{ in } K_0$$

Note also that the character is an element of K_0/G_0

<u>Lemma 7.11</u> If $\mathcal{C} \simeq \mathcal{D}$, then $K_0(\mathcal{C}) \simeq K_0(\mathcal{D})$ and $G_0(\mathcal{C}) \simeq G_0(\mathcal{D})$ as rings if $\mathcal{C}$ + $\mathcal{D}$ are $\otimes$-equivalent

<u>Proof</u> This follows since $[X:S_i]$, $(X:I_j)$ are invariant under $\simeq$ (as well as #simples/iso #indecomp./iso)

Example – $K_0(\mathrm{Vect}^\omega(G)) \simeq G_0(\mathrm{Vect}^\omega(G)) \simeq \mathbb{Z}[G]$ for any ω. However, $\mathrm{Vect}^\omega(G)$ and $\mathrm{Vect}^{\omega'}(G)$ need not to be equivalent

– Recall that $A = \mathbb{F}_5[\mathbb{Z}/5\mathbb{Z}]$-mod had its inde indec by Jordan blocks

$$\begin{matrix} (1) & \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \\ S_0 = I_0 \simeq \mathbb{F}_5 & I_1 & I_2 & I_3 & P_0 = I_5 \simeq A \ A\text{-proj} \end{matrix}$$

Thus, since $AA \simeq A \oplus A \oplus A \oplus A \oplus A$ we have $\underset{\text{rank } 1}{K_0(A\text{-proj})} \simeq \underset{\text{rank } 1}{G_0(A\text{-mod})} \simeq \mathbb{Z} \not\simeq \underset{\text{rank } 5}{K_0(A\text{-mod})}$

Here is a final observation for now:

Lemma 7.12 If $\mathcal{C}$ is braided, the $K_0(\mathcal{C})$ and $G_0(\mathcal{C})$ are commutative.

Proof Being braided implies that $XY \cong YX$, so $[X][Y] = [XY] = [YX] = [Y][X]$

Example

This lemma makes it transparent why $\mathrm{Vect}^{\omega}(G)$ can not be braided unless G is commutative