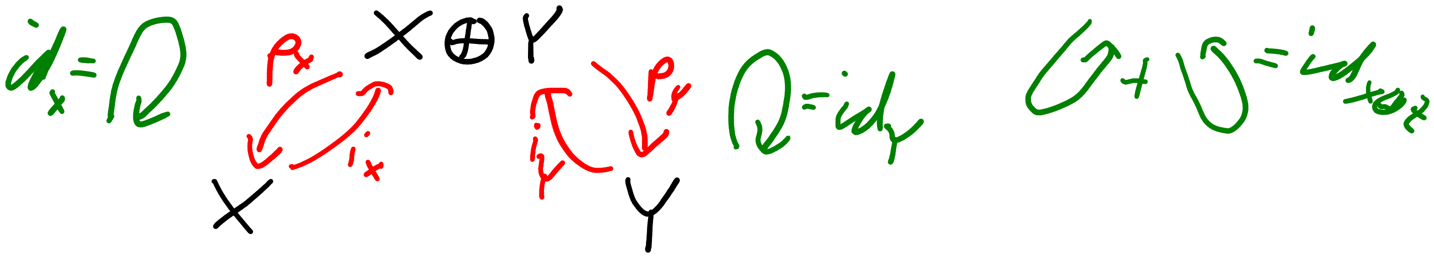
Lecture 6 Additive, linear and abelian cats - Def+Ex

Any target of a topological invariant should be computable, e.g. within the realm of linear algebra

Def 6.1 A cat $\mathcal{C}$ is called additive (or $\oplus$-cat) if:

- $Hom_{\mathcal{C}}(X,Y)$ is a $\mathbb{Z}$-module (an abelian group) and $\circ$ is biadditive
- $\exists\, 0 \in \mathcal{C}$ (zero object) such that $Hom_{\mathcal{C}}(0,0)=0$
- $\forall X,Y\ \exists Z$ such that $\exists\, p_X, p_Y, i_X, i_Y$ such that

unique up to unique iso and denoted $X\oplus Y$
direct sum



Def 6.2 A functor $F: \mathcal{C} \to \mathcal{D}$ between add cats is called additive if $\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ is a morphism of $\mathbb{Z}$-modules.

This looks like the wrong def, but it is the correct one:

Lemma 6.3 (Proof omitted) For any additive functor there exists a natural iso $F(X) \oplus F(Y) \simeq F(X \oplus Y)$

Dropping the second and third condition of $\mathcal{C}$ to be add ($\exists$ zero, $\exists \oplus$) we get the notions of a $\mathbb{Z}$-linear cat. Or, similarly ($\mathbb{k}$-) linear cats.

We thus get: **Lemma 6.4** Linear functors between add linear cats are additive

Here is the reason why this generalizes linear algebra:

<u>Proposition 6.5</u> Every linear cat $\mathcal{C}$ can be embedded in an additive cat $Mat(\mathcal{C})$, i.e. $\mathcal{C} \hookrightarrow Mat(\mathcal{C})$ linear

<u>Proof</u>: $Mat(\mathcal{C})$, called the matrix envelope, is defined as follows.

- Objects Formal direct sums $X_1 \oplus \dots \oplus X_m$ of $X_i \in \mathcal{C}$
- Morphisms: $f : X_1 \oplus \dots \oplus X_m \longrightarrow Y_1 \oplus \dots \oplus Y_n$

$$f = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$$

$f_{ij} \in Hom_{\mathcal{C}}(X_i, Y_j)$

- Composition is matrix multiplication

e.g. $f = \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix}$ $\quad g = \begin{pmatrix} g_{11} & g_{21} \end{pmatrix}$

$f : X_1 \oplus X_2 \to Y_1 \oplus Y_2$ $\qquad g : Y_1 \oplus Y_2 \longrightarrow Z$

makes sense in $\mathcal{C}$

$$g \circ f = \begin{pmatrix} g_{11} & g_{21} \end{pmatrix} \circ \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix} = (g_{11} \circ f_{11} + g_{21} \circ f_{12} ,\; g_{11} \circ f_{21} + g_{21} \circ f_{22})$$

$\mathcal{C}$ is linear

Now $\mathcal{C} \hookrightarrow Mat(\mathcal{C})$

$X \longmapsto X \qquad f \longmapsto f$

This is "as good as it could be": it doesn't hurt to assume $\mathcal{C}$ to be additive

- It's easy, but we can now compute with matrices.
- If $\mathcal{C}$ is $\otimes$, rigid, pivotal, braided or whatever, then so is $Mat(\mathcal{C})$, basically in a unique way

e.g. $(X_1 \oplus X_2)(Y_1 \oplus Y_2) := X_1Y_1 \oplus X_1Y_2 \oplus X_2Y_1 \oplus X_2Y_2$

Examples – Set is neither linear nor additive
– $\mathrm{Vect}_{\mathbb{k}}$ and $\mathrm{fdVect}_{\mathbb{k}}$ are both linear and add.
Their skeletons $\mathrm{Mat}_{\mathbb{k}}$ and $\mathrm{fdMat}_{\mathbb{k}}$ are linear, and additive and are of course the blueprint models of additive cats.

– $\mathrm{Vect}(M)$ is linear, but not additive.
$\mathrm{Mat}(\mathrm{Vect}(M))$ is equiv to the category of M-graded $\mathbb{k}$ VS
– The full subcategory $\mathrm{evenVect}_{\mathbb{k}} \subset \mathrm{fdVect}_{\mathbb{k}}$ of even dimensional vector spaces is linear and additive

If one wants to do homological algebra cat., which is more involved than linear algebra, then one also needs a more complicated notion:

Def 6.6 Let $\mathcal{C}$ be additive. A kernel of $f: X \to Y$ is an object $\mathrm{Ker}(f) \in \mathcal{C}$ together with a morphism $k: \mathrm{Ker}(f) \longrightarrow X$ such that

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & \\
 & \mathrm{Ker}(f) \arrow[u, "k"] \arrow[r, "0"'] & Y \\
K' \arrow[uur, bend left, "k'"] \arrow[ur, dashed, "\exists!"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

commutes, i.e. $k \circ f = 0$

Abuse of notation: Usually only write $\mathrm{Ker}(f)$

A cokernel $\mathrm{coker}(f)$ is a kernel in $\mathcal{C}^{op}$

If kernel/cokernels exist, then they are unique (up to isomorphism)

Examples – $\mathrm{Vect}_{\mathbb{K}}$ has (co)kernels (and so has $\mathrm{fdVect}_{\mathbb{K}}$) $\forall$ with $\ker(f)$ = usual kernel and cokernel = usual cokernel

– $\mathrm{evenVect}_{\mathbb{K}}$ does have neither kernels nor cokernels: A matrix of odd rank between even dim VS has kernel of odd dim.

Def 6.7 A linear, add cat having kernels and cokernels is called **abelian** if for every $f: X \to Y$ $\exists$ a sequence

$$\mathrm{Ker}(f) \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} \mathrm{coker}(f)$$

- $j \circ i = f$
- $(I, i) = \mathrm{coker}(k)$
- $(I, j) = \ker(c)$

$\uparrow$ I = called the Image

$\uparrow$ called canonical decomposition of f

Most invariants of classical top. take values in abelian categories

In abelian categories we have:

- kernels, cokernels, images
- subobjects $Y \subset X$, i.e. $Y \xrightarrow{f} X$ such that $\ker(f) = 0$
- quotients $X \twoheadrightarrow Y$, i.e. $X \xrightarrow{f} Y$ such that $\mathrm{coker}(f) = 0$
- subquotients, i.e. subobjects of quotients
- For $Y \overset{f}{\subset} X$ one can define $X/Y := \mathrm{coker}(f)$

All of these are exactly what you think they are:

Theorem 6.8 (Mitchell embedding) $\forall\, \mathcal{C}$ abelian $\exists\, A$ algebra such that $\mathcal{C} \hookrightarrow A\text{-mod}$ as linear, $\oplus$ cats

fully faithful $\nearrow$

Examples / Remarks - The previous theorem is pretty useless in general since A is neither unique nor are such A easy to compute

- The name comes from the fact that abelian categories generalize abelian groups

- $\mathrm{Vect}_{\mathbb{K}}$ and $\mathrm{fdVect}_{\mathbb{K}}$ are abelian

- There exist the notion of an abelian envelope, i.e. an abelian cat $Ab(\mathcal{C})$ such that $\mathcal{C} \hookrightarrow Ab(\mathcal{C})$

However $Ab(\mathcal{C})$ is far away from being "nice", eg $Ab(\mathcal{C})$ need not to be $\otimes$ even if $\mathcal{C}$ is.

$\Rightarrow$ Homological algebra is much harder than linear algebra

In ∞-dim "hell breaks loose", so:

Def 6.9 An object $X \neq 0 \in \mathcal{C}$ is called *simple*, if

$Y \subset X \Rightarrow Y = 0$ or $Y = X$ "The atoms"

$X \neq 0$ is said to have *finite length*, if $\exists$

$$0 \subset X_0 \subset \ldots \subset X_n = X \leftarrow \text{Jordan-Hölder (JH) filtration}$$

such that X_i / X_{i-1} is simple.

Theorem 6.10 JH filtrations, if they exist, are unique up to potential reordering.

In particular, {iso classes and # of simples appearing in a JH filt of X} is independent of the chosen JH

Example – In $\mathrm{Vect}_{\mathbb{K}}$ and $\mathrm{fdVect}_{\mathbb{K}}$, the ground field $\mathbb{K}$ is the only simple object. In particular, $\mathbb{K}^{\mathbb{N}}$ is not of finite length, but all $X \in \mathrm{fdVect}_{\mathbb{K}}$ are

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Def 6.11 $0 \neq X \in \mathcal{C}$ is called indecomposable if $X \simeq Y \oplus Z \Rightarrow Y = 0$ or $Z = 0$ "Also atoms"

Theorem 6.12 (Krull-Schmidt)

If X has finite length, then

$$X \simeq X_1 \oplus \dots \oplus X_n \quad \leftarrow \text{indecomposable (inde)}$$

and such a decomposition is unique up to isomorphism (aka reordering)

Examples / Remarks – Note that simple $\underset{\not\Leftarrow}{\Rightarrow}$ inde.

- In $\mathrm{Vect}_{\mathbb{K}}$ the only inde. is $\mathbb{K}$ itself, so simple $\Leftrightarrow$ inde

- In $\mathrm{Vect}^{\oplus}_{\mathbb{K}}(M) := \mathrm{Mat}(\mathrm{Vect}_{\mathbb{K}}(M))$, which is abelian, the simples $= M =$ inde.

- Same with twists ω

- $\mathrm{mod}(A)$ or $\mathrm{comod}(C)$ for algebras/algebras in $\mathcal{C}$ are abelian.

- Let $A = \mathbb{F}_2[\mathbb{Z}/2\mathbb{Z}]$. Then A-mod is abelian with only simple object $\mathbb{F}_2$, but $A \in A$-mod itself is inde., not simple

To see this, note that $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is the action matrix of σ ($\mathbb{Z}/2\mathbb{Z}=\langle 1,\sigma\rangle$) on A, which, after base change, is $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ ← an honest Jordan block

$\to A/\mathbb{F}_2$

In particular, $0\subset\mathbb{F}_2\subset A$ is the JH filt.

Lemma (Schur) Let $X\in\mathcal{C}$ be simple and $Y\in\mathcal{C}$ be indec. Then $\mathrm{End}_{\mathcal{C}}(X)\cong\mathbb{K}$ and $\mathrm{End}_{\mathcal{C}}(Y)$ is local

"all endos are isos"

"all endos are either isos or nilpotent"

Example With A as above we have $\mathrm{End}_{\mathcal{C}}(\mathbb{F}_2)=\mathbb{F}_2$, but
$\mathrm{End}_{\mathcal{C}}(A)\cong\mathbb{K}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix}\right)\cong\mathbb{F}_2[X]/X^2$ where

$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}2&2\\2&2\end{pmatrix}=0$ in char 2

Here is the right notion of a functor $F : \mathcal{C} \to D$ between abelian categories:

Def 6.13 A sequence $\dots \to X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \to \dots$ is called **exact**, if $\operatorname{im}(f_{i-1}) = \ker(f_i)$.

F is called **exact** if every exact sequence $X \to Y \to Z$ is mapped to an exact seq. $F(X) \to F(Y) \to F(Z)$

An equivalence of abelian cats uses exact functors

Def 6.14 $P \in \mathcal{C}$ is called **projective** if $\operatorname{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \to \mathrm{Vect}_{\mathbb{K}}$ is exact

$I \in \mathcal{C}$ is called **injective** if I is projective in $\mathcal{C}^{op}$ "dual notions"

Def 6.15 We say $\mathcal{C}$ is **finite abelian** if $\exists A$ fd algebra such that $\mathcal{C} \simeq A\text{-fdmod}$ as abelian cats

Crucial facts about $\mathcal{C}$ finite:

- Hom spaces are fd
- # simples / iso $= n < \infty$ $\{S_i\}$
- # pim / iso $= n < \infty$ $\{P_i\}$
- # iim / iso $= n < \infty$ $\{I_j\}$
- JH and Krull-Schmidt hold
- $\mathrm{Proj} \subset \mathcal{C}$ are linear + additive subcategories
  $\mathrm{Inj} \subset \mathcal{C}$
- $\mathrm{Proj} = \mathrm{Inj} = \mathcal{C} \Leftrightarrow$ (simple $\Leftrightarrow$ inde)

pim = projective + indecomposable
iim = injective inde.

Proj = proj objects
Inj = inj objects

The Cartan Matrix $C(\mathcal{C})$ can thus be defined by counting $[P_i : S_j]$

mult of S_j in P_i via JH

Def 6.16 Let $M \in \mathcal{C}$, $\mathcal{C}$ finite. The the character of M is $ch(M) = \bigoplus_{\text{simples } S_i} [M : S_i] S_i$ seen as an element of the Grothendieck group of $\mathcal{C}$.

Example Again, let $A = \mathbb{F}_2[\mathbb{Z}/2\mathbb{Z}]$ and let $\mathcal{C} = A\text{-fdmod}$. Then $\mathbb{F}_2$ is simple, but not projective and A is prim.

$ch(A) = 2\mathbb{F}_2 = ch(\mathbb{F}_2 \oplus \mathbb{F}_2)$, but $A \not\cong \mathbb{F}_2 \oplus \mathbb{F}_2$

$C(\mathcal{C}) = (2)$ as A is the only prim module in $\mathcal{C}$. Moreover, A is also inin

In this case there are also no other inde, but that is a coincidence (In general there might even be ∞-many inde.)

Example Let $A = \mathbb{F}_5[\mathbb{Z}/5\mathbb{Z}]$ and $\mathcal{C} = A\text{-fdmod}$

Then the Jordan blocks $\langle\sigma\rangle$ (action of σ)

$\begin{pmatrix} 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$
$\updownarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\updownarrow$
$\mathbb{F}_5$	indec $\notin$ Proj	indec $\notin$ Proj	Indec $\notin$ Proj	A
$\notin$				$\in$
Proj				Proj $=$ Inj
$\mathrm{ch} = 1 \cdot \mathbb{F}_5$	$\mathrm{ch} = 2 \cdot \mathbb{F}_5$	$\mathrm{ch} = 3 \cdot \mathbb{F}_5$	$\mathrm{ch} = 4 \cdot \mathbb{F}_5$	$\mathrm{ch} = 5 \cdot \mathbb{F}_5$

and $C(\mathcal{C}) = (5)$, $\mathrm{Proj} = \mathrm{Inj} \subsetneq \mathcal{C}$