Lecture 5 Braided cats - Defs, examples, graphics

Def 5.1 A braided category $\mathcal{C}$ is a $\otimes$-cat together with $\tau = \{\tau_{X,Y} : XY \xrightarrow{\sim} YX\}$ such that

↑ a choice

Today all $\mathcal{C}$'s are braided

- naturality $\tau_{W,X}(fg) = (gf)\tau_{Y,Z}$
- compatibility $\tau_{X,YZ} = (id_Y\, \tau_{X,Z}) \circ (\tau_{X,Y}\, id_Z)$

$$\tau_{XY,Z} = (\tau_{X,Z} \otimes id_Y) \circ (id_X \otimes \tau_{Y,Z})$$

$\mathcal{C}$ is further called symmetric if

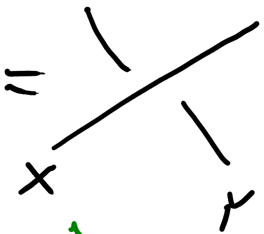
↑ a property

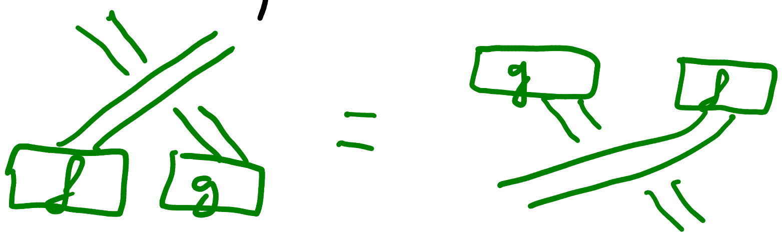
$$\tau_{Y,X}\,\tau_{X,Y} = id_{XY} \quad \forall X, Y \in \mathcal{C}$$

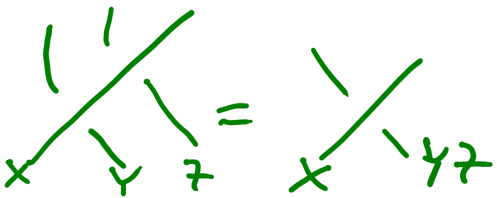
Def 5.2 A braided functor $F:\mathcal{C}\to\mathcal{D}$ is a functor such that

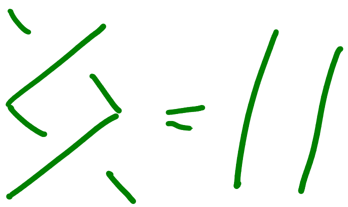
$$F(T^{\mathcal{C}}_{X,Y}) = T^{\mathcal{D}}_{F(X),F(Y)}$$

⇝ We get the notion of equivalences of braided cats etc.

We draw: $T_{X,Y} =$ 

⇝ naturality 

⇝ compatibility 

– sym.  ← bad for knot theory :-)

Examples - $(\mathrm{Sets}, \times)$ is symmetric

- $(\mathrm{Vect}_{\mathbb{K}}, \otimes)$ is symmetric
- $\mathrm{Vect}^{\omega}(M)$ is braided iff M is commutative

(if M is not commutative, then $mn \not\cong nm$)

In case M is commutative, then braidings are in 1:1 correspondence with $c: M \times M \to \mathbb{K}^*$ such that

$$c(gh, j) = c(g,j)\, c(h,j)\, \omega(g,h,j)^{-1}\, \omega(h,g,j)\, \omega(h,j,g)^{-1}$$

$$c(g, hj) = c(g,h)\, c(g,j)\, \omega(g,h,j)\, \omega(g,j,h)^{-1}\, \omega(j,g,h)$$

$\Rightarrow$ Classification of braidings (this is very hard beyond these toy cases)

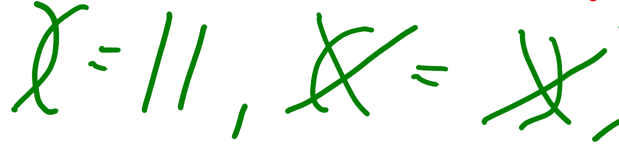
Examples

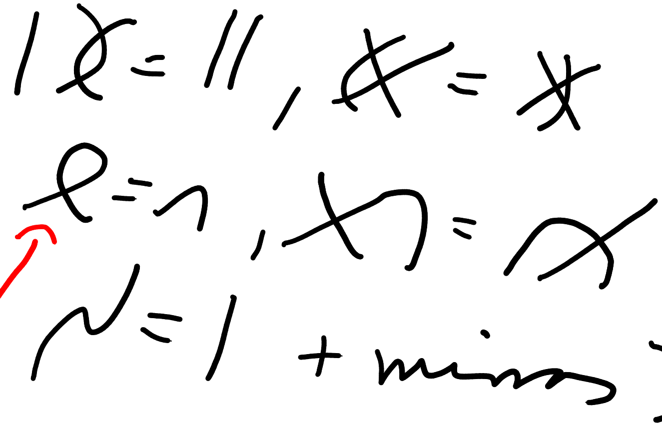
- In particular, for $M=1$, $\mathrm{Vect}(1) \simeq \mathrm{fdVect}_{\mathbb{K}}$ allows only one braiding
- More general, if $\omega = 1$, then c = bilinear pairing which classify the braidings. These are symmetric iff c is symmetric
- If $M = \mathbb{Z}/2\mathbb{Z} = \langle 1, \sigma \rangle$, $\omega = 1$, then $c = 1$ defines a symmetric braiding.
If $c(\sigma,\sigma) = -1$, the one gets a non-equi braiding and $(\mathrm{Vect}(\mathbb{Z}/2\mathbb{Z}), \tau^c)$ is known as super vector spaces.

Examples

- The category Sym is the free symmetric cat. gen. by one object $\bullet$

This is the braiding, of course

Recall: $\mathrm{Sym} = \langle \bullet, \times \mid$  $\rangle$

- The category TL only has an interesting braiding if linearized
- The category $\mathrm{Br} = \langle \bullet, \times, \cap, \cup \mid$  + mirrors $\rangle$ is symmetric and pivotal

Called twist

Its quantum version $q\mathrm{Br}$ is braided and pivotal

Br is the free symmetric pivotal cat gen. by one self-dual object and trivial twist

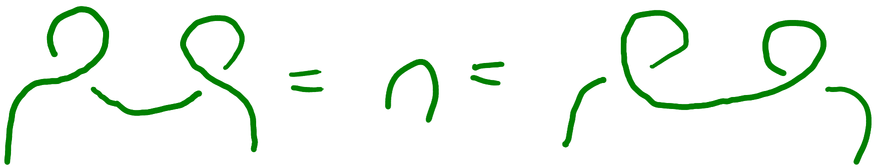
From now on: $\mathcal{C}$ is braided and pivotal

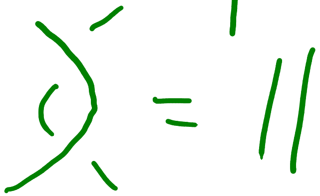
Theorem 5.3 (Reidemeister, unoriented)

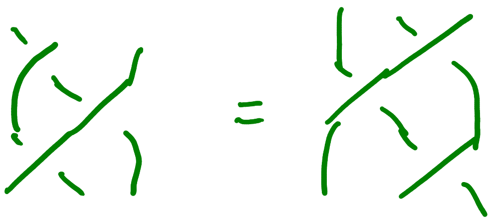
The free braided pivotal cat generated by one self dual object $\bullet$ has the presentation

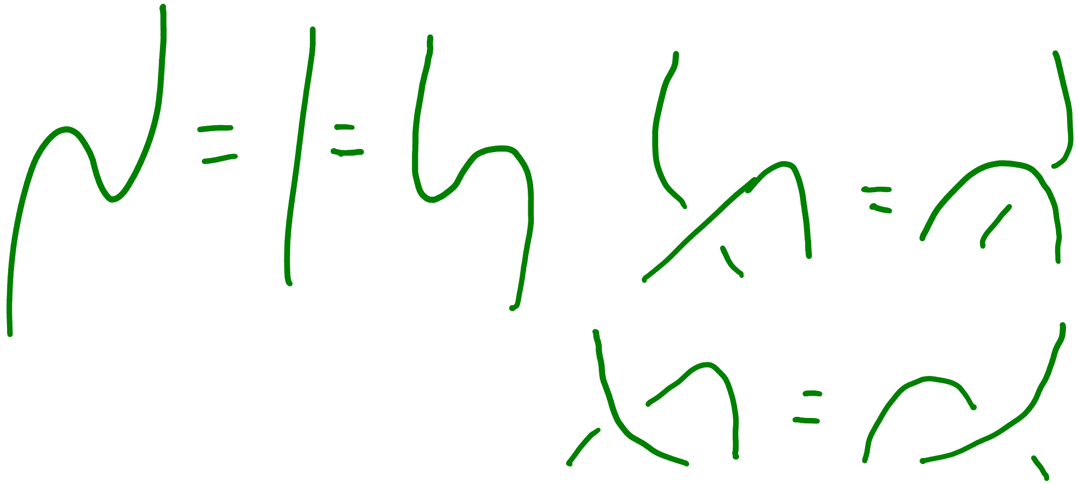
$$f1Tan = \langle \bullet, X, \cup, \cap \mid \text{Reidemeister calc} \rangle$$

"f = framed"

fR1: 

R2: 

R3: 

Iso: 

"framed 1-tangles"

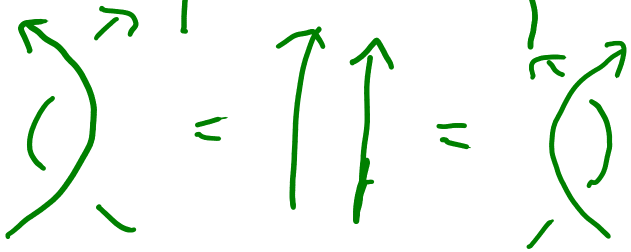
Theorem 5.4 (Oriented version)

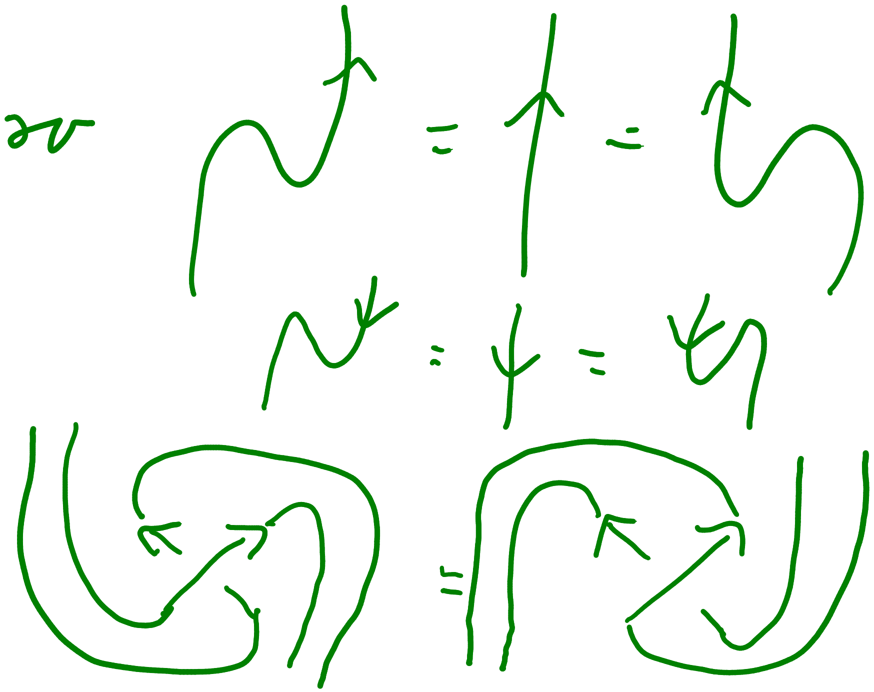
The free braided pivotal cat generated by one object + (⇝ ↑) has the presentation

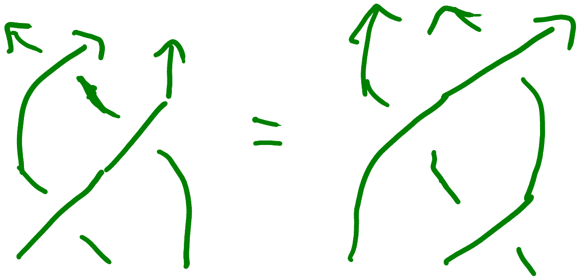
$f\ell Tan^{or} = \langle +,$  $|$ or. Reidemeister rel $\rangle$

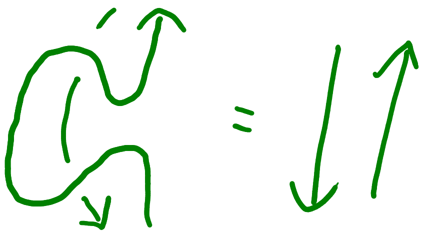
"oriented, framed 1 tangle"

$f R1$ 

$R2$ 

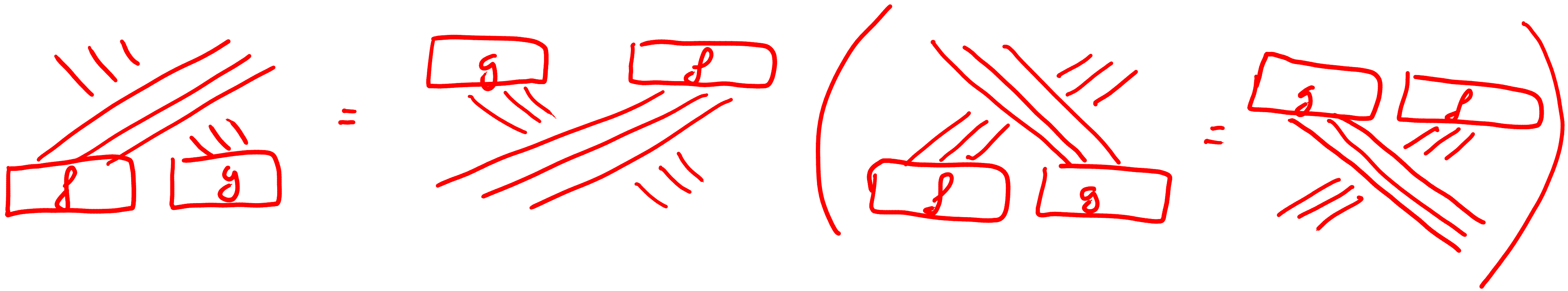
Z or 

$R3$ 

$R2'$: 

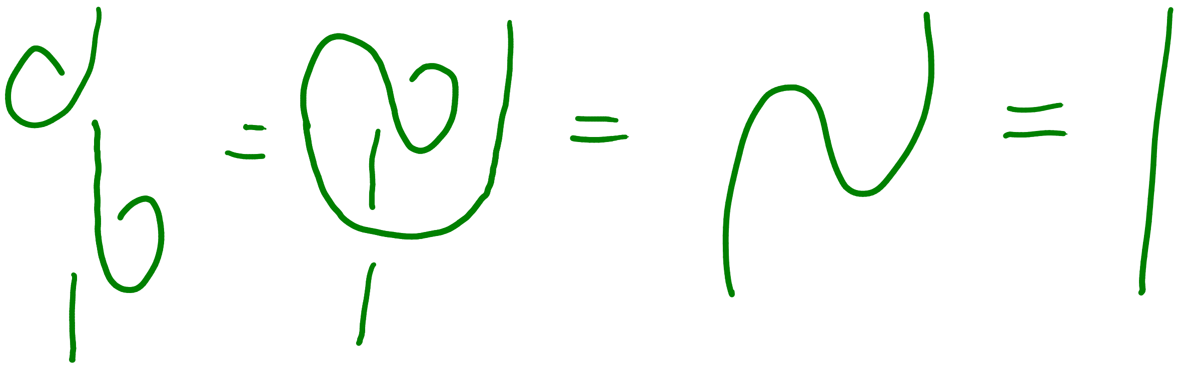
Remark

Formulated in other words, together with

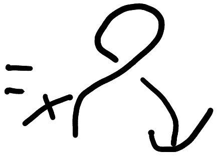


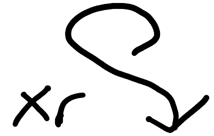
Every braided pivotal category has a diagram calculus given by oriented, framed tangles

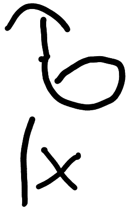
Of course, there are some consequences e.g.

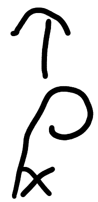


Def 5.5

The **positive twists** θ_x^+ are $\theta_x^+ =$ 

The **negative twists** θ_x^- are $\theta_x^- =$ 

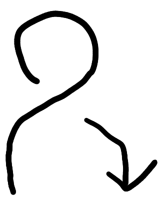
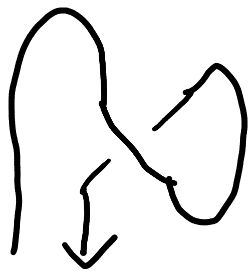
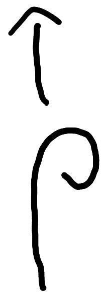
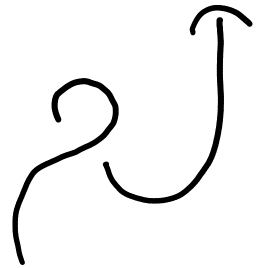
The halfmates of twists are $\tilde{\theta}_x^+ =$ 

$\tilde{\theta}_x^- =$ 

Lemma 5.6

Twists and their halfmates are the same data

Proof.

 =  and  = 

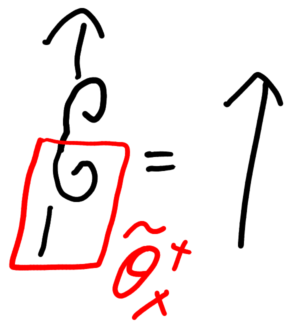
Lemma 5.7

$\tilde{\theta}_X^+$ and $\tilde{\theta}_X^-$ are natural isomorphism $id_X \Rightarrow id_X$

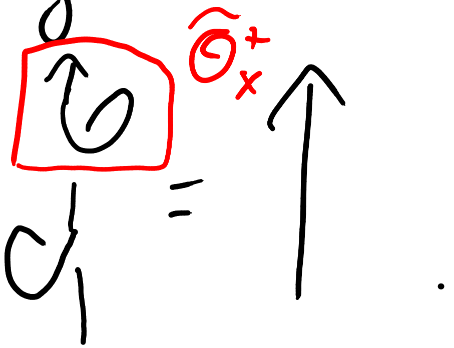
Moreover, $(\hat{\theta}_X^+)^* = \tilde{\theta}_{X^*}^-$ and $(\hat{\theta}_X^-)^* = \tilde{\theta}_{X^*}^+$

Proof

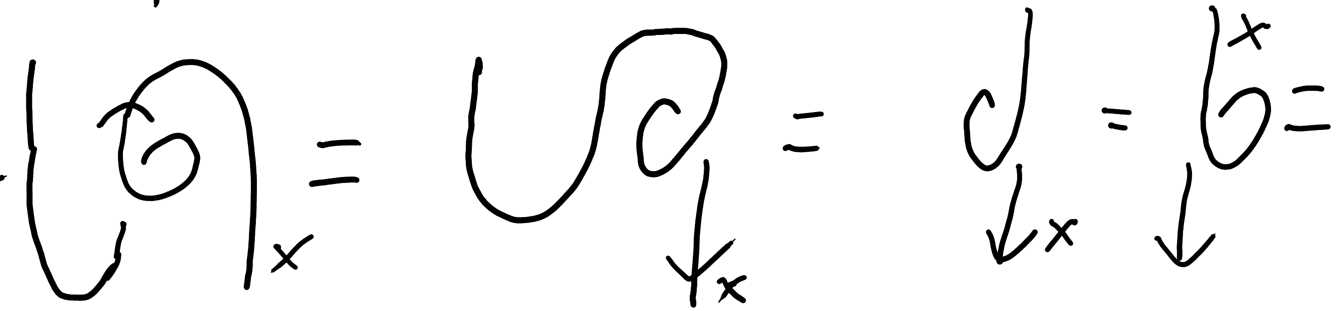
By the above, we have the equa.



and



Now

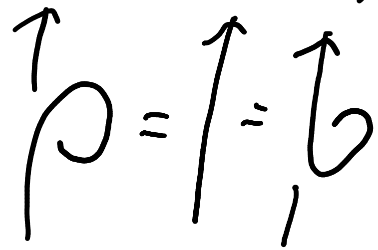


$= \theta_{X^*}^-$

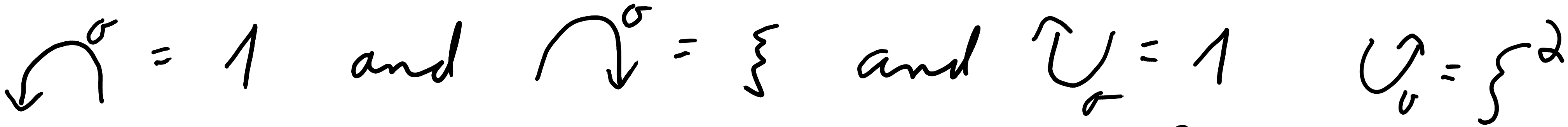
Similarly for θ_X^-.

Examples

- In $\mathrm{fdVect}_{\mathbb{K}}$ the pairing is symmetric, thus

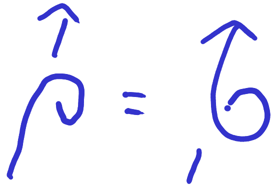


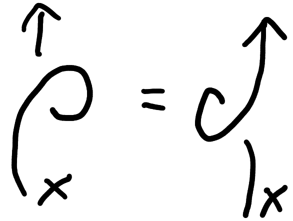
- Recall that for $G = \mathbb{Z}/3\mathbb{Z} = \langle 1, \sigma, \sigma^2 \rangle$ we can choose a pivotal structure



Thus, for the trivial braiding we have

$\Rightarrow$ twist are not trivial and also not equal

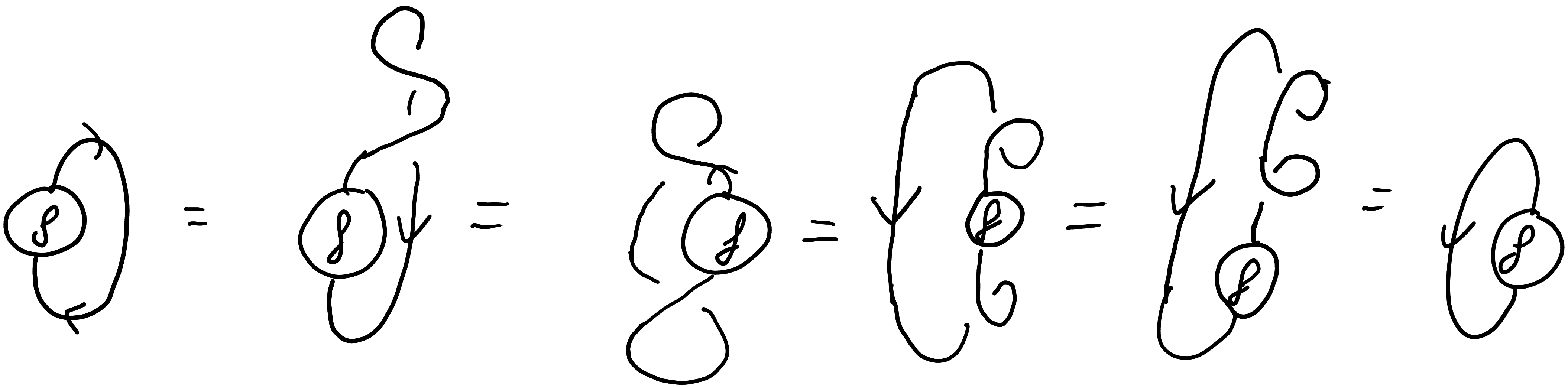
However  is an important property motivating:

Def 5.8 $\mathcal{C}$ is called **ribbon** if  $\forall x$

Lemma 5.9

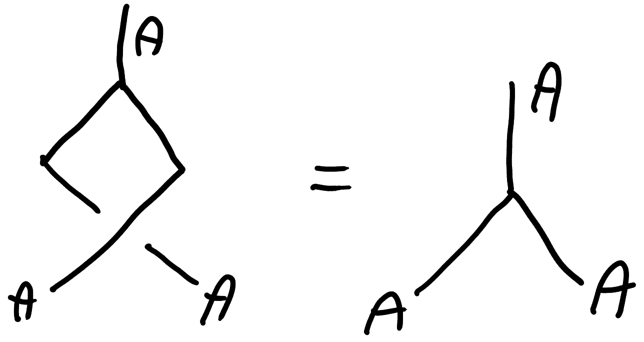
Ribbon categories are spherical

Proof



Here is the generalization of a comm. algebra

Def. 5.10 An algebra $A \in \mathcal{C}$ ← needs only braided
is called **commutative** if



Similarly for coalgebras.

Examples

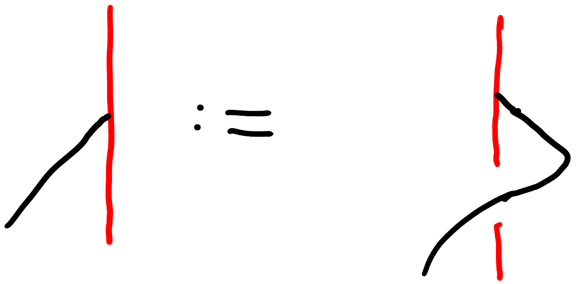
- In $\mathrm{Vect}_{\mathbb{K}}$ comm algebras are comm algebras
- In $\mathrm{Vect}(\mathbb{Z}/2\mathbb{Z})$ with the super braiding, comm algebras are supercommutative algebras, i.e. $\mathbb{Z}/2\mathbb{Z}$-graded algebras such that $xy = (-1)^{\deg x \deg y} yx$

Proposition 5.11 Let $M \in \mathcal{C}$ be a right A-module object in $\mathcal{C}$ ← only braided
↑ comm. alg

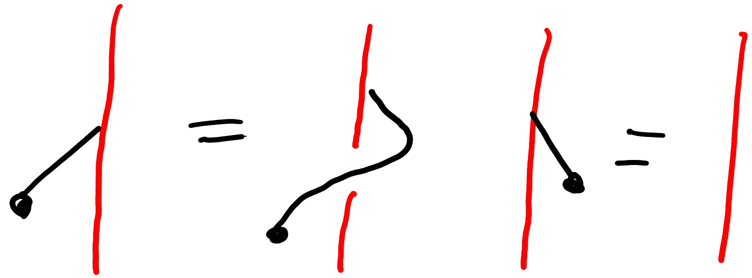
Then M is also a right module object.

In particular, $\mathrm{Mod}_{\mathcal{C}}(A) \simeq (A)\mathrm{Mod}_{\mathcal{C}} \simeq (A)\mathrm{Mod}_{\mathcal{C}}(A)$

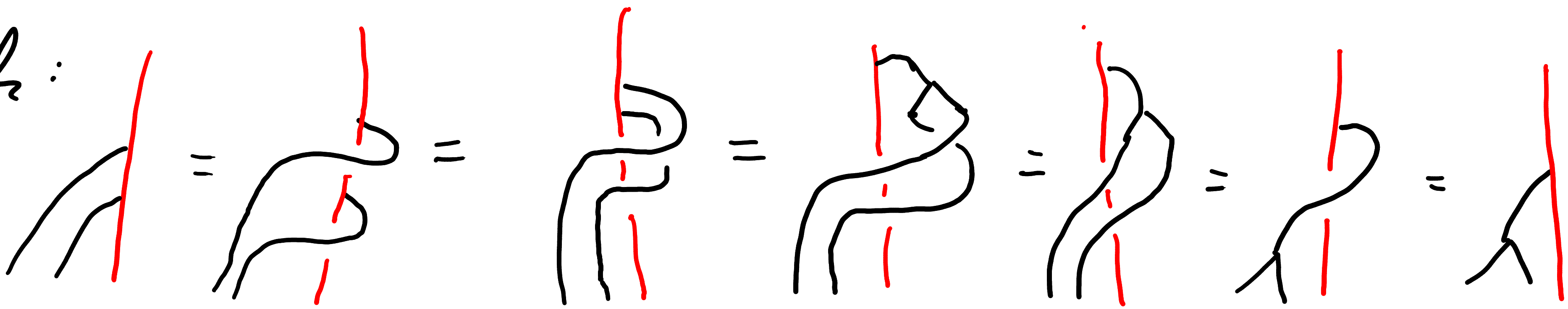
Proof: Define



Check:



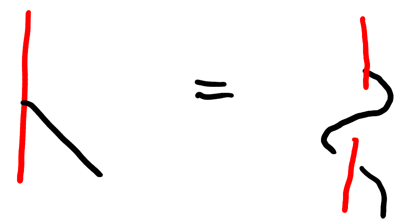
Check:



This implies that $\mathrm{Mod}_{\mathcal{C}}(A)$ is a $\otimes$-cat.

One can say more:

<u>Proposition 5.12</u> The full subcategory $\mathrm{Mod}^{\circ}_{\mathcal{C}}(A) \subset \mathrm{Mod}_{\mathcal{C}}(A)$ given by A-modules satisfying



is braided with braiding inherited from $\mathcal{C}$.

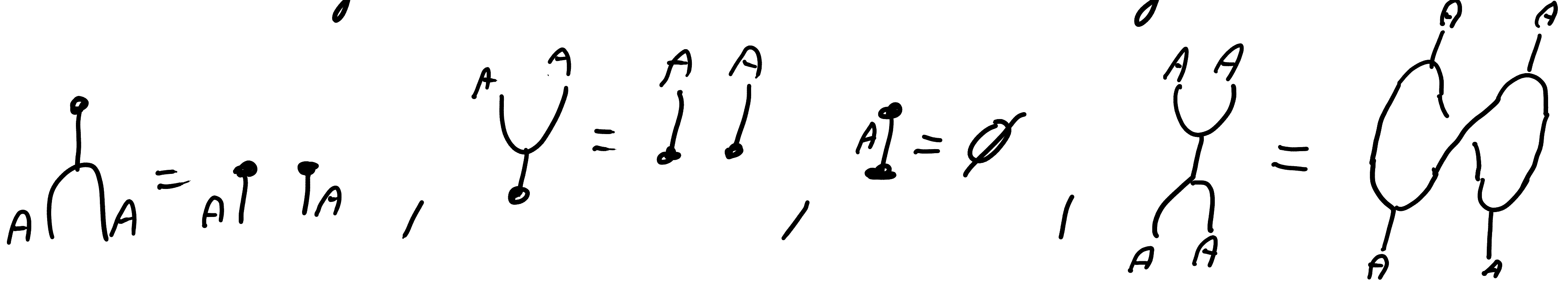
Proof: Diagram considerations

Example

If $\mathcal{C}$ is symmetric, then $\mathrm{Mod}^{\circ}_{\mathcal{C}}(A) = \mathrm{Mod}_{\mathcal{C}}(A)$, which is thus also symmetric.

We can also generalize bialgebras

Def 5.13 A **bialgebra** $A \in \mathcal{C}$ ← only braided
is an algebra in $\mathcal{C}$ and a coalgebra in $\mathcal{C}$ such that



Examples

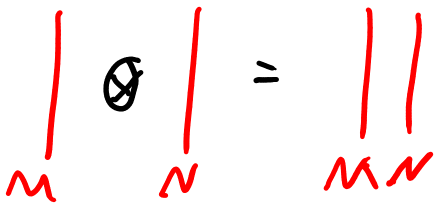
- Bialgebras in $\mathrm{Vect}_{\mathbb{K}}$ are bialgebras
- Bialgebras in $\mathrm{Vect}(\mathbb{Z}/2\mathbb{Z})$ with the super braiding are super bialgebras

There is the evident notion of a bialgebra morphism etc.

Proposition 5.14 Let $B \in \mathcal{C}$ be a bialgebra. The category $\mathrm{Mod}_{\mathcal{C}}(B)$ is monoidal.

Proof

Here is a way to define $M N$ for $M, N \in \mathrm{Mod}_{\mathcal{C}}(B)$:



, action

One can now check using the usual yoga, that this defines a $\otimes$-structure.