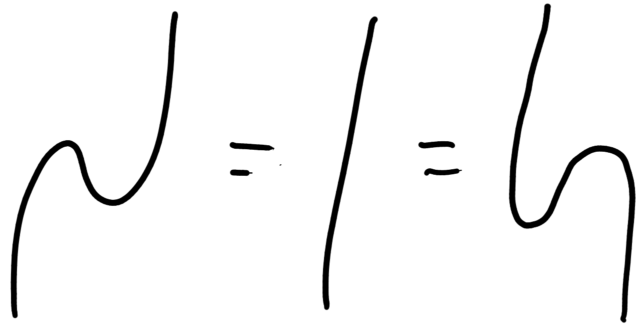
Lecture 4 Pivotal cats - defs, examples + diagrams

Recall A monoidal category $\mathcal{C}$ admits a graphical calculus with upwards planar isotopies.

However in some examples we have seen that we want to allow "change of orientation" e.g. in TL

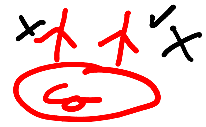


Next How to do this in general

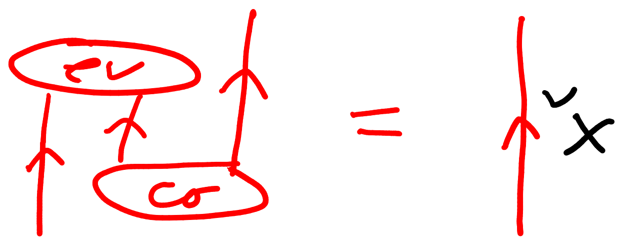
Def 4.1 A left dual ${}^\vee X \in \mathcal{C}$ of $X \in \mathcal{C}$ is an object together with two morphisms

This is a choice: Beware

$ev_X : {}^\vee X X \longrightarrow \mathbb{1}$ $\qquad$ $coev_X : \mathbb{1} \longrightarrow X X^\vee$



such that they are non-degenerate i.e.



- left duals are unique, if they exist
- similarly right duals $X^\vee$, $\widetilde{ev}_X : X X^\vee \to \mathbb{1}$
  $\widetilde{coev}_X : \mathbb{1} \to X^\vee X$

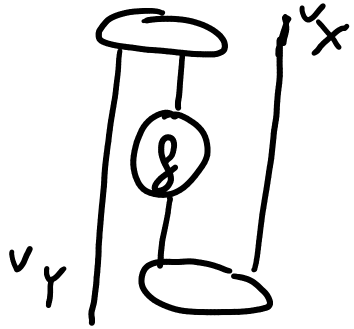
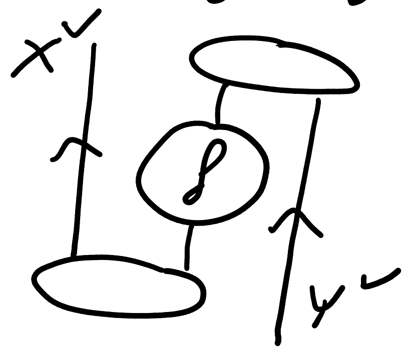
Def 4.2 A $\otimes$-cat $\mathcal{C}$ is called **rigid** if every object has a left and a right dual

Examples

- $(\mathrm{Vect}_{\mathbb{K}}, \otimes)$ is not rigid, but its $\otimes$-subcategory $\mathrm{fdVect}_{\mathbb{K}}$ is. The left = right dual is the dual vector space
- $\mathrm{Vect}^{\omega}(M)$ is rigid iff M = a group and ${}^{\vee}m = m^{\vee} = m^{-1}$
- TL rigid with ${}^{\vee}\bullet = \bullet^{\vee} = \bullet$

From now on $\mathcal{C}$ is rigid

For $f: X \to Y$ we can define its **mates** ${}^{\vee}f, f^{\vee}$ via

${}^{\vee}f =$  and $f^{\vee} =$ 

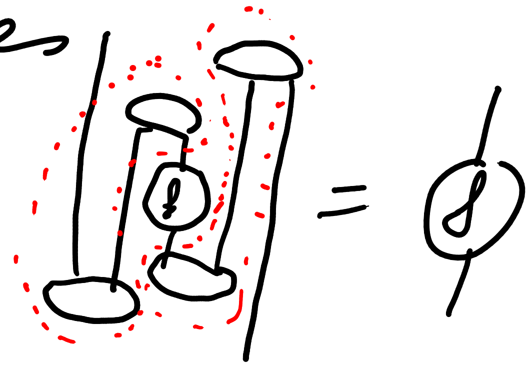
One can check that ${}^\vee(g\circ f) = {}^\vee f \circ {}^\vee g$ and $(g\circ f)^\vee = f^\vee \circ g^\vee$

which gives: ${}^\vee(gf) = {}^\vee f\, {}^\vee g$ and $(gf)^\vee = f^\vee g^\vee$

Proposition 4.3 There are duality functors ${}^\vee -, -^\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{coop}}$

These are monoidal equivalences

Proof: Observe $({}^\vee f)^\vee = f = {}^\vee(f^\vee)$, eg



So we have quasi-inverses $-^\vee, {}^\vee - : \mathcal{C}^{\mathrm{coop}} \to \mathcal{C}$

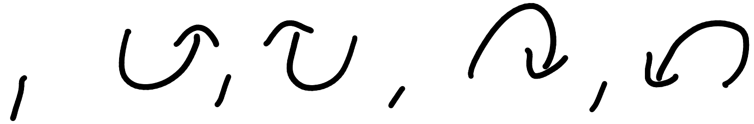
In particular, ${}^\vee(X^\vee) \simeq X \simeq ({}^\vee X)^\vee$

Example | Warning We do not have ${}^\vee X \simeq X^\vee$ in general, which would imply $X^{\vee\vee} \simeq X \simeq {}^{\vee\vee}X$

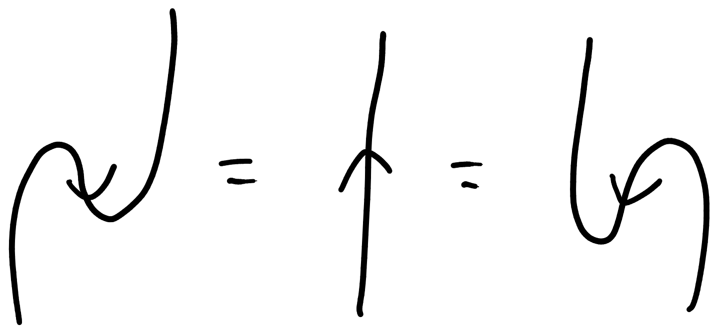
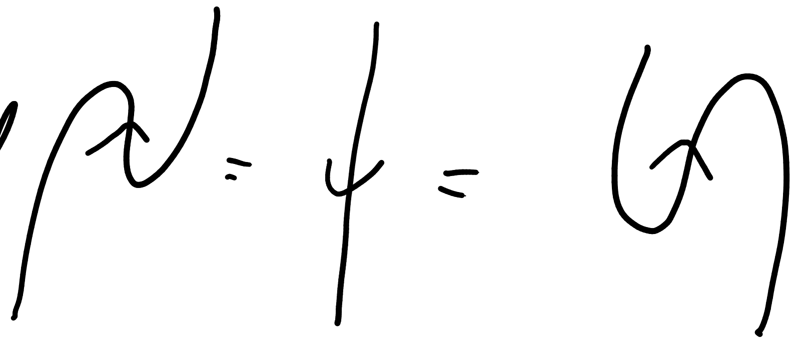
Examples of such need some background (or are trivial....), eg. fd comodules over $u_q(sl_2)$ is an example

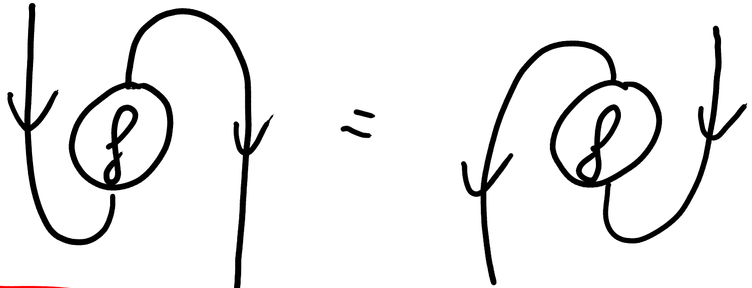
Def 4.4 A pivotal cat $\mathcal{C}$ is a rigid cat such that $^\vee -, -^\vee$ are isomorphic. If we fix such an isomorphism, then we call this a pivotal structure.

In other words $^\vee X \simeq X^\vee$ fixed, or $X \simeq X^{\vee\vee}$ fixed or $X \simeq {}^{\vee\vee}X$ fixed. Thus, we can write X^* without ambiguity. We will see $\simeq$ diagrammatically later.

Graphical: $X\uparrow$, $\uparrow X^* =: \downarrow X$, 

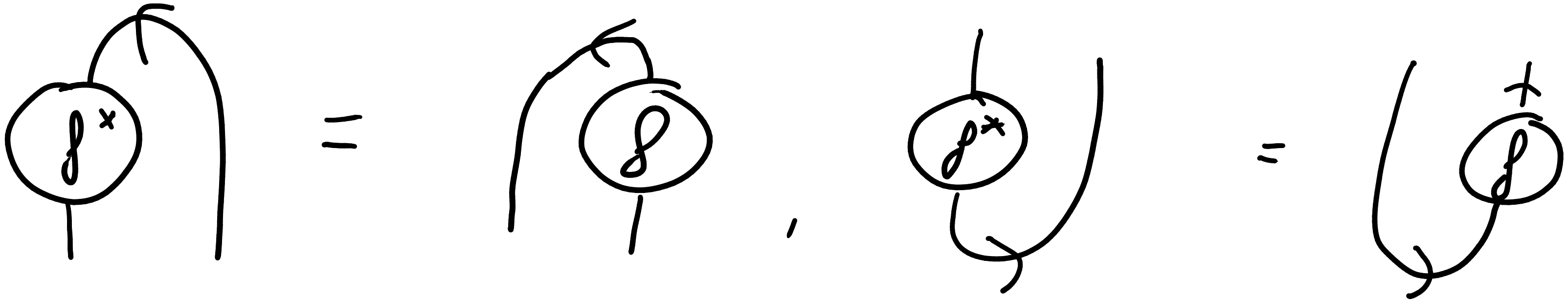
and

 and 

$f^* =$ 

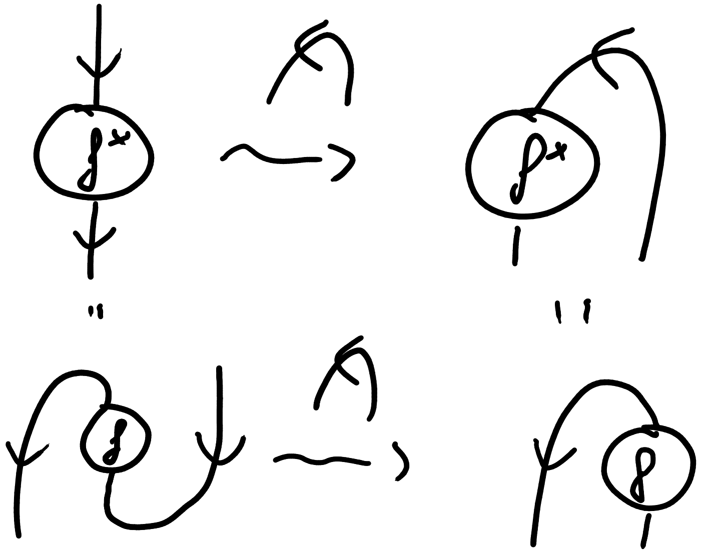
Let $\mathcal{C}$ now be pivotal.

Proposition 4.5 We have the sliding rules



+ orientation reversals

Proof:  are invertible operations e.g.



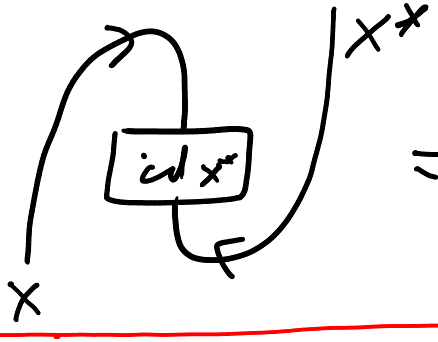
Proposition 4.6 We have

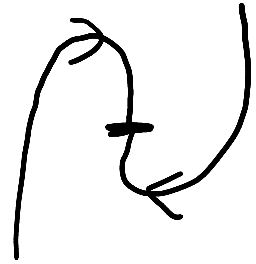
$\mathrm{Hom}_{\mathcal{C}}(XY, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, ZY)$ etc.

Proof Exactly the same argument.

Define $\psi_X : X \longrightarrow X^{**}$ via

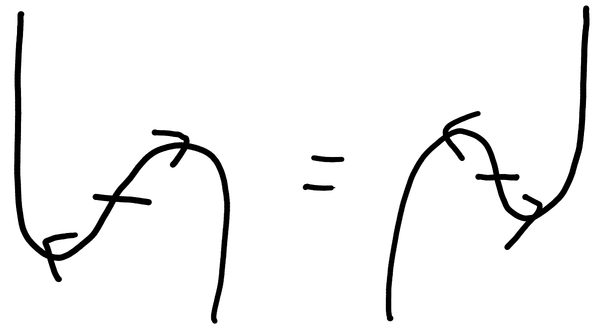
A pivotal structure

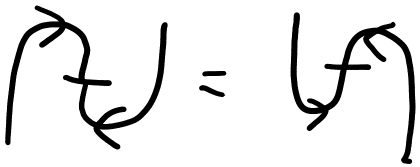
$\psi_X :=$ 

$=:$ 

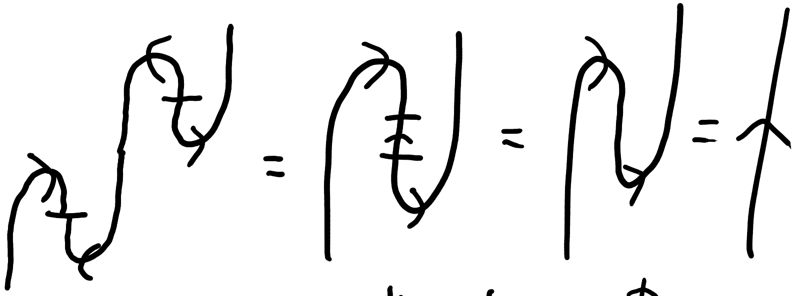
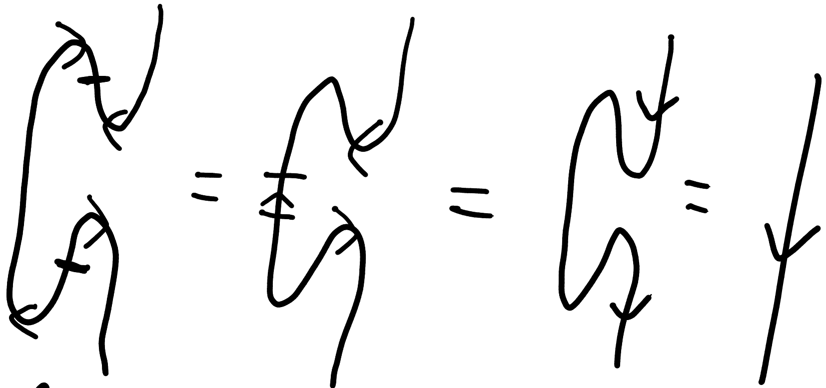
ψ_{X^*} is the dual of X^*; so X^{***}

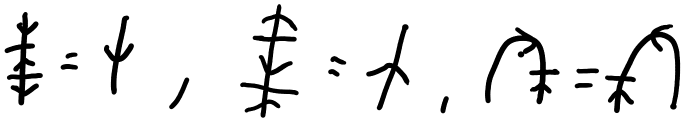
Lemma 4.6

ψ_X is invertible with inverse 

Thus, also .

Proof

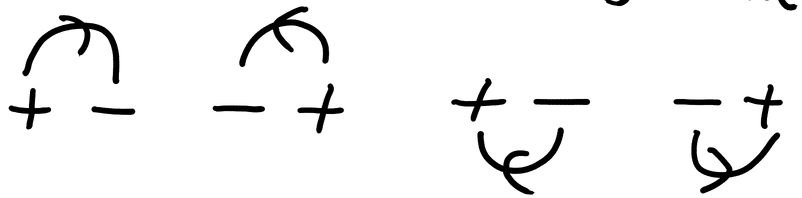
 and 

 etc.

Free examples

- The free rigid category generated by one object $\bullet$ has two additional objects ${}^\vee\bullet$, $\bullet^\vee$ which satisfy $({}^\vee\bullet)^\vee = \bullet = {}^\vee(\bullet^\vee)$, but nothing else. In particular ${}^{\vee\vee}\bullet \neq \bullet \neq \bullet^{\vee\vee}$

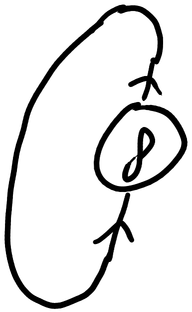
- The free pivotal category generated by one object $\bullet =: +$ has its dual $\bullet^* =: -$ and morphisms gens

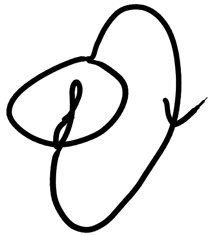


- The free pivotal category generated by one self-dual object $\bullet = \bullet^*$ is TL

no orientations

Def 4.7 The left $tr^l_{\mathcal{C}}(f)$ and right $tr^r_{\mathcal{C}}(f)$ trace of $f: X \to X$ are

$tr^l_{\mathcal{C}}(f) =$  $\in End_{\mathcal{C}}(\mathbb{1})$

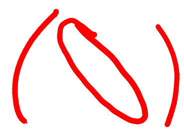
$tr^r_{\mathcal{C}}(f) =$  $\in End_{\mathcal{C}}(\mathbb{1})$

This is a vast generalization of the case $\mathcal{C} = \mathrm{fdVect}_{\mathbb{k}}$:

$X = \mathbb{k}\{x_1, \dots, x_n\}$  $: X^* X \to \mathbb{k}$  $: \mathbb{k} \to X^* X$

$X^* = \mathbb{k}\{x_1^*, \dots, x_n^*\}$ $\quad x_i^* x_j \mapsto \delta_{ij}$ $\quad \mathbb{1} \longmapsto \sum x_i^* x_i$

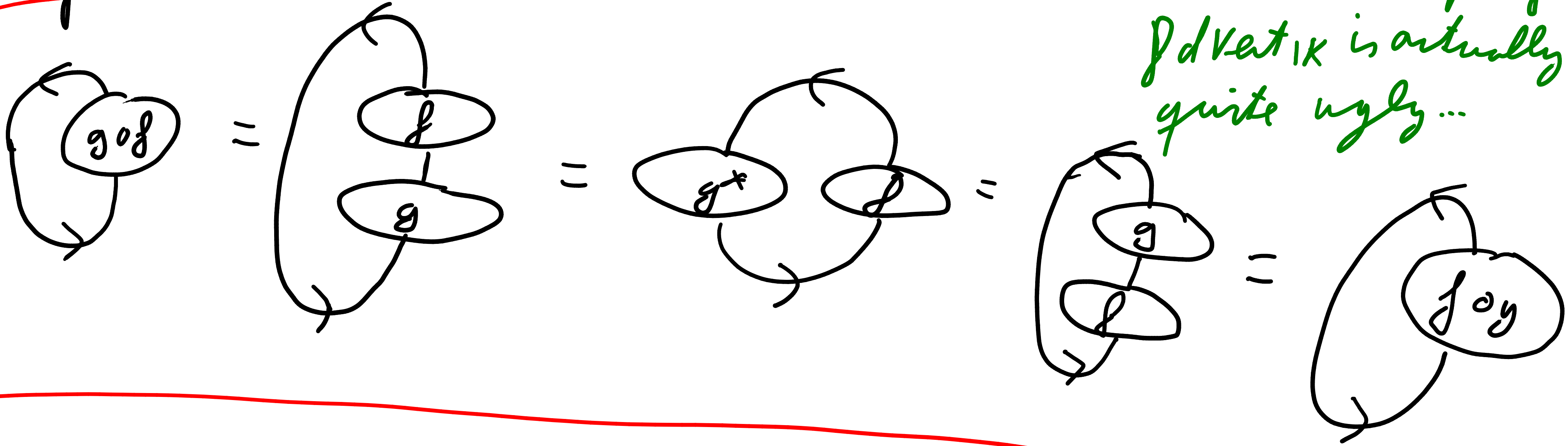
Then $tr^l_{\mathcal{C}}(f) = tr(f) = tr^r_{\mathcal{C}}(f)$

↑ classical trace $f =$ 

Lemma 4.8 Traces are symmetric, eg. $tr^l_e(g\circ f) = tr^l_e(f\circ g)$

The matrix proof in $fdVect_{\mathbb{K}}$ is actually quite ugly...

Proof

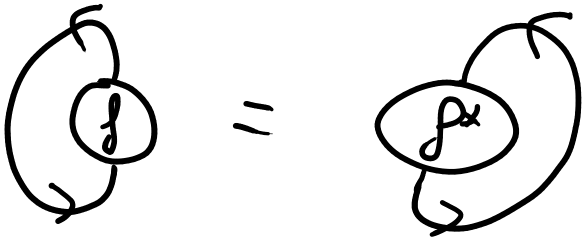


Lemma 4.9 Traces are linear, i.e. $tr^l_e(\alpha) = tr^r_e(\alpha)$ and $tr^l_e(\alpha\cdot f) = \alpha\cdot tr^l_e(f)$ etc., where $\alpha\in End_e(\mathbb{1})$

Proof α is a flying bubble

Lemma 4.10 We have $tr^l_e(f) = tr^r_e(f^*)$ etc.

Proof



Motivated by $tr\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix} = tr(id_X) = \dim X$ we also have

Def 4.11 $\dim^l_e(X) = tr^l_e(id_X)$ and $\dim^r_e(X) = tr^r_e(id_X)$

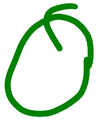
left dimension — right dimension

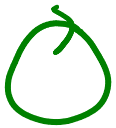
Clearly $\dim^L_e(X) = \dim^R_e(X^*) = \dim^L_e(X^{**})$

$\dim^R_e(X) = \dim^L_e(X^*) = \dim^R_e(X^{**})$

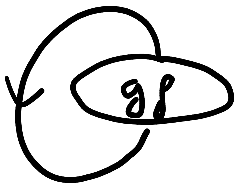
$\dim^L_e(\mathbb{1}) = \dim^R_e(\mathbb{1}) = id_{\mathbb{1}} = \emptyset$

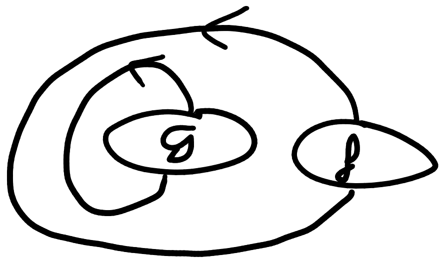
Picture

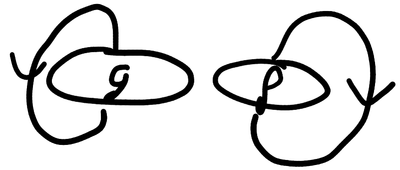
$\dim^L_e =$ 

$\dim^R_e =$ 

A pivotal category (with $tr^l_e(f) = tr^r_e(f) =: tr_e(f)$ $\forall f$) is called **spherical** because its diagrams live on a sphere $\Rightarrow$

$tr_e(gf) =$ 

$=$ 

$=$ 

$= tr_e(g)\, tr_e(f)$

Examples - of course, $\mathcal{C} = \mathrm{fdVect}_{\mathbb{K}}$ is spherical and $tr_{\mathcal{C}} = tr$, $\dim_{\mathcal{C}} = \dim$

- in the generic TL, which is spherical, we have $\dim_{\mathcal{C}}(\bullet) = \bigcirc \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$
- $\mathrm{Vect}^{\omega}(G)$ is pivotal.
- For any Hopf algebra H, H-Rep is rigid can be chosen to be
  fd left module
  $H = \mathbb{K}$ is the example above
- A special case of the above, $H = \mathbb{K}(G)$ is spherical and said that $\dim_{\mathcal{C}}(V)$ is the usual dim, but in $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \mathbb{K}$.
  In particular, $\dim_{\mathcal{C}}(V) = 0$ is entirely possible.
  ≠ 0
- For quantum enveloping algebras $\dim_{\mathcal{C}}$ is a quantum number can be chosen to be ← will give a link invariant

Remark By def. rigid $\Leftarrow$ pivotal $\Leftarrow$ spherical

The converses are not true in general.

However, as I will discuss in a later lecture, under certain reasonable assumptions "$\Rightarrow$" are conjectured to hold.

Still, after ~30 years

So it should be too easy to find examples of "$\not\Rightarrow$".

We have already seen that rigid $\not\Rightarrow$ pivotal.

Also pivotal $\not\Rightarrow$ spherical:

One can choose a pivotal structure on $\mathrm{Vect}^{\omega}(G)$ which is in general not spherical (More later)

Observation: Any $\otimes$-functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ carries an object X having a left dual to an object having a left dual. Similarly, of course for right duals. However, the following is the correct notion of a rigid functor:

Def 4.12 Let $\mathcal{C}, \mathcal{D}$ be rigid cats. A monoidal functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is called rigid if $\mathcal{F}(ev_X^{\mathcal{C}}) = ev_X^{\mathcal{D}}$ etc. hold.

Recall that we fixed these.

$\rightsquigarrow$ We get the notion of rigid/pivotal/spherical cats to be equi.

Bigger example

Take $G = \mathbb{Z}/2\mathbb{Z} = \langle 1, \sigma \rangle$ w the non-trivial 3-cocycle.

Then $\mathrm{Vect}^\omega(G)$ is pivotal with two different choices of ev, coev, but always $\sigma^* = \sigma$:

- We can let $ev_\sigma = 1$, $\widetilde{ev}_\sigma = -1$
  $coev_\sigma = -1$ and $\widetilde{coev}_\sigma = 1$

So we get $\dim_\epsilon^\ell(\sigma) = 1 = \dim_\epsilon^r(\sigma)$

- We can let $ev_\sigma = 1$, $\widetilde{ev}_\sigma = 1$, $coev_\sigma = -1$
  and $\widetilde{coev}_\sigma = -1$.

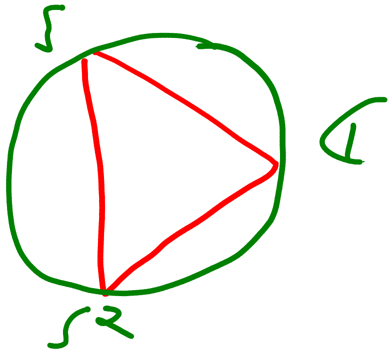
Hence, $\dim_\epsilon^\ell(\sigma) = -1 = \dim_\epsilon^r(\sigma)$

Bigger example 2

Take eg $\omega = \mathrm{triv}$

Take $G = \mathbb{Z}/3\mathbb{Z} = \langle 1, \sigma, \sigma^2 \rangle$, ω any cocycle and let $\lambda_g = \omega(1,1,g)\,\omega(g,g^{-1},g)\,\omega(g,1,1) \in \mathbb{K} = \mathbb{C}$

Let $d : G \to \mathbb{C}^*$, $d(1) = 1$, $d(\sigma) = \xi$, $d(\sigma^2) = \xi^2$



Then one can check that $ev_\sigma = 1$, $\widetilde{ev}_\sigma = d(\sigma)\lambda_\sigma$, $coev_\sigma = \lambda_\sigma^{-1}$, $\widetilde{coev}_\sigma = d(\sigma)^{-1}$.

This gives

$$\dim_e^l(\sigma) = 1 \cdot d(\sigma)^{-1} = \xi^2 \neq \xi = d(\sigma)\lambda_\sigma\lambda_\sigma^{-1} = \dim_e^r(\sigma)$$

$\Rightarrow$ not spherical

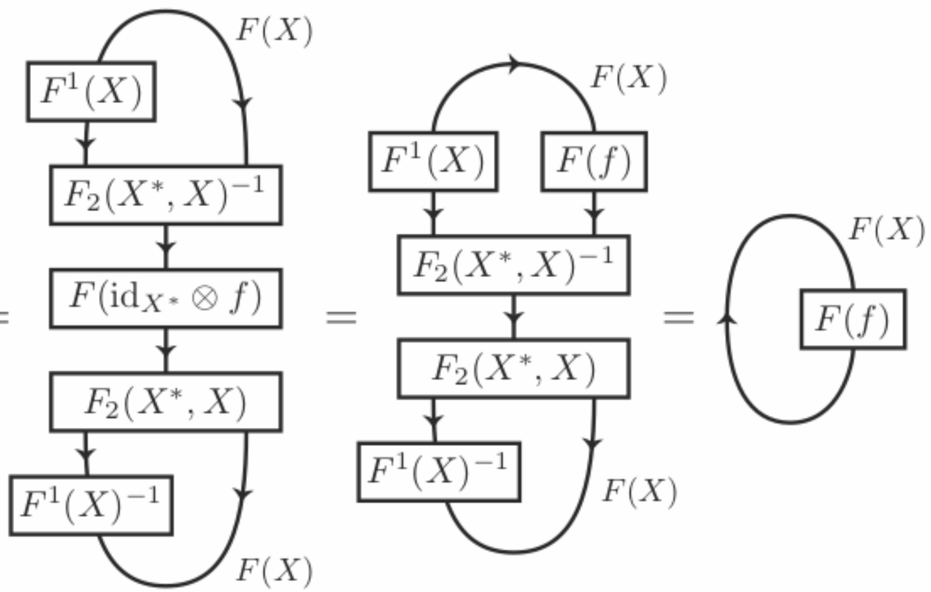
Lemma 4.13 Let $F : \mathcal{C} \to \mathcal{D}$ be a rigid functor between pivotal cats. Then:

$$F(\mathrm{tr}^{\ell}_{\mathcal{C}}(f)) = \mathrm{tr}^{\ell}_{\mathcal{C}}(F(f))$$

$$F(\mathrm{tr}^{r}_{\mathcal{C}}(f)) = \mathrm{tr}^{r}_{\mathcal{C}}(F(f))$$

Proof An omitted diagrammatic argument, eg:

$F_0^{-1}F(\mathrm{tr}_l(f))F_0 =$

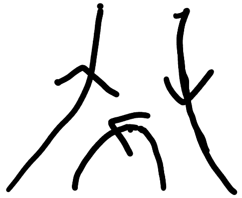


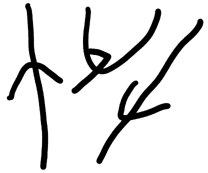
$= \mathrm{tr}_l(F(f)).$

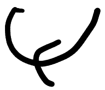
Lemma 4.14

The object XX^* is a Frobenius object in $\mathcal{C}$. Similarly for X^*X.

Proof

$m : XX^*XX^* \longrightarrow XX^*$ 

$\Delta =$ 

$\iota :$ 

$\varepsilon :$ 

Checking the Frobenius condition is easy, eg