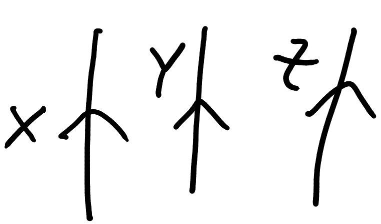
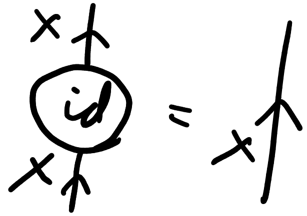
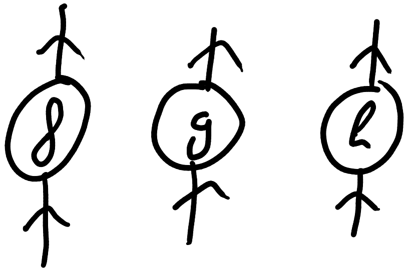
Lecture 3 Monoidal cats II – more diagrams

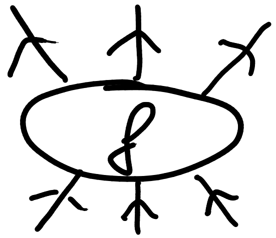
Recall $\mathcal{C}$ (strict) monoidal allows:





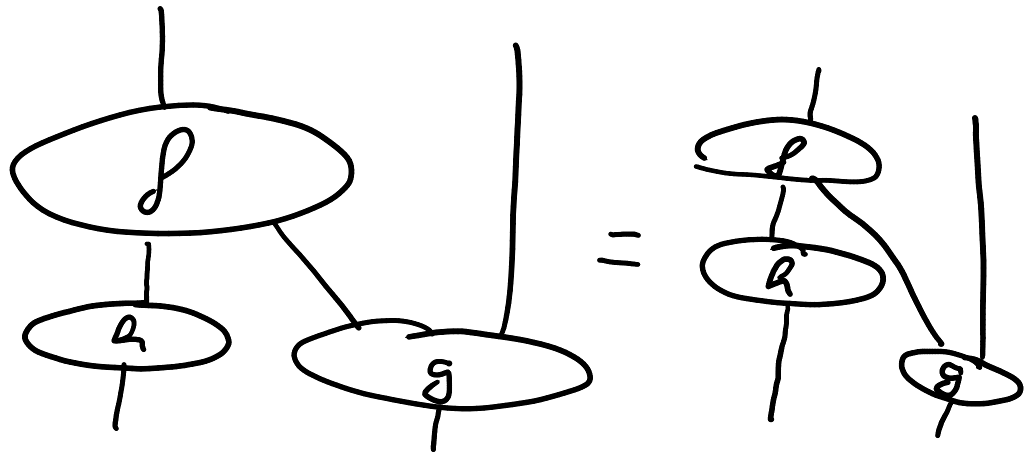
$\mathbb{1} = \emptyset$ etc.





etc.

Planar isotopies respecting ↑ eg.



How to construct new cats from old ones?

- $\mathcal{C}^{op}$ ← opposite $\circ$ composition
- $\mathcal{C}^{co}$ ← opposite $\otimes$ composition
- $\mathcal{C}^{coop}$ ← opposite $\circ$ and $\otimes$ compositions
- $\mathrm{End}_{\otimes}(\mathcal{C})$ = cat of all $\otimes$ functors $\mathcal{C} \to \mathcal{C}$
- Quotient cats $\mathcal{C}/\sim$

Congruence on all $\mathrm{Hom}_{\mathcal{C}}(X,Y)$

Question: How to construct $\otimes$-cats using graphical calculus?

Reminder A monoid M is generated by a set $\{m_j\}$ if every element in M can be written as a product $m_{i_1} \cdots\cdots m_{i_k}$

As a monoid; all monoid axioms are implied

A monoid M has generators-relations $\langle W | R \rangle$ if M is generated by W and two words in the alphabet W are the same in M iff they can be obtained from one another via operations from R

Example $- \mathbb{N}^2 \simeq \langle x, y \mid xy = yx \rangle$

concrete elements (m,n)

$(1,0) \leftarrow\!\shortmid x$

$(0,1) \leftarrow\!\shortmid y$

gen-rel elements

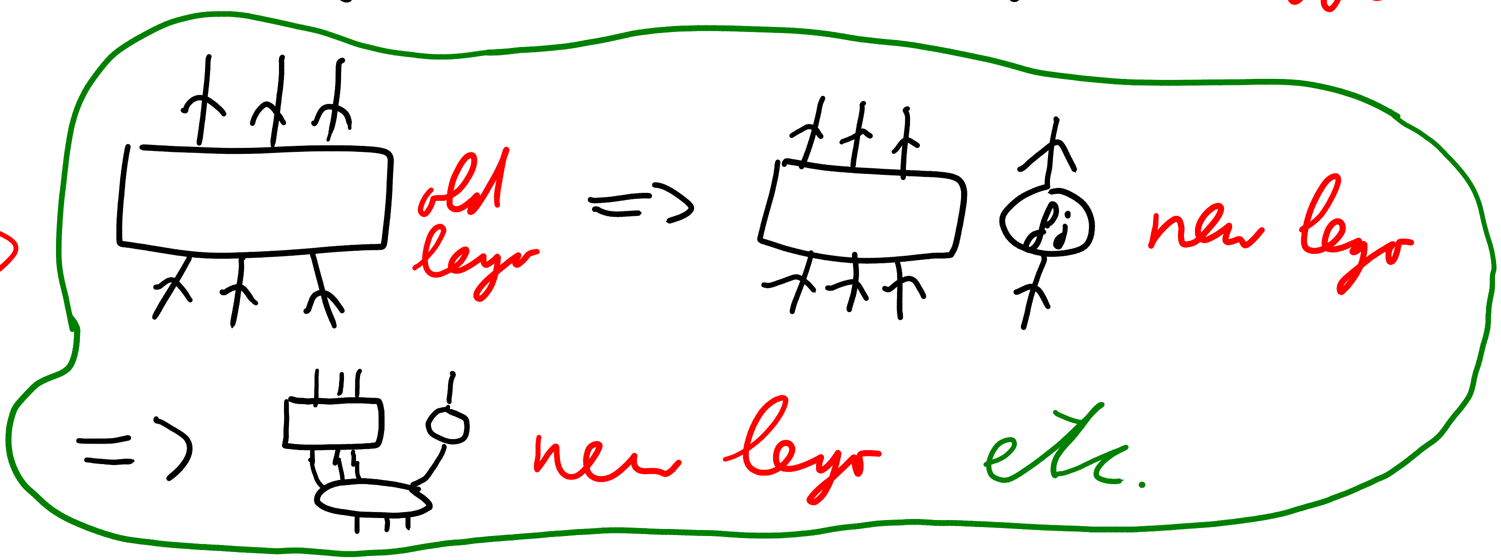
$xyxxy =_R x^3 y$

Def 3.1

We say a $\otimes$-cat $\mathcal{C}$ is **$\otimes$-generated** by a set of objects $\{X_i\}$ and morphisms $\{f_j\}$ if:

- Every $X \in \mathcal{C}$ is of the form $X_{i_1} \ldots X_{i_n}$
- Every $f \in \mathcal{C}$ is a successive composition, either $\circ$ or $\otimes$, of morphisms from $\{f_j\} \cup \{\mathrm{id}_{X_i}\}$

Lego principle

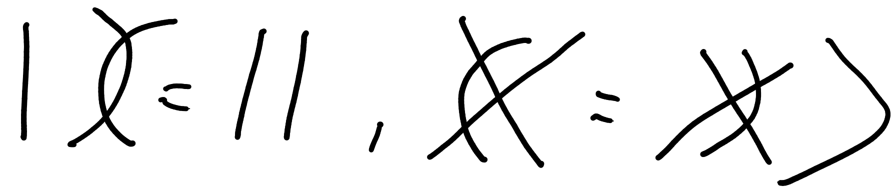


Def 3.2

We say a $\otimes$-cat $\mathcal{C}$ has a generator-relation presentation $\langle X, W \mid R \rangle$ if

- X $\otimes$-generates $Ob(\mathcal{C})$
- W $\circ$-$\otimes$-generates $Mor(\mathcal{C})$
- R consists of congruences such that two $\circ$-$\otimes$-words in W are the same in $\mathcal{C}$ iff they can be obtained from one another by applying congruences from R

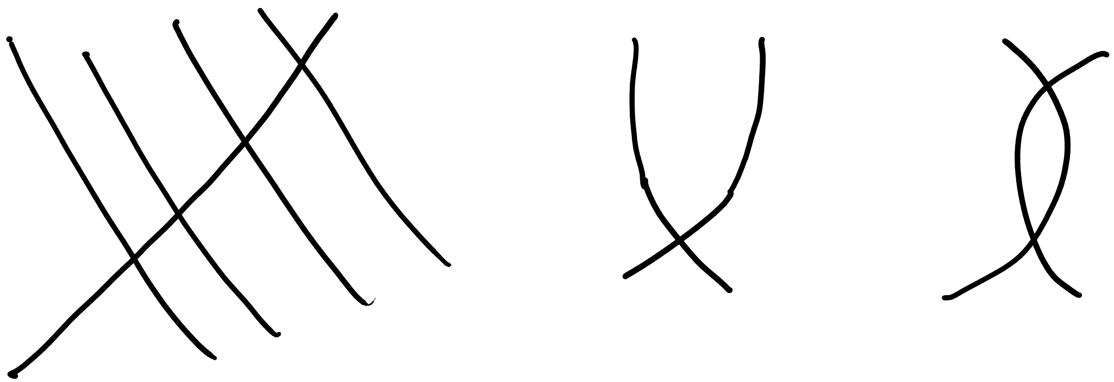
Example "All symmetric groups at once"

$\mathrm{Sym} = \langle \bullet, X : \bullet \otimes \bullet \to \bullet \otimes \bullet \mid$  $\rangle$

Now:

Objects are of the form $\bullet \; \bullet \; \bullet \; \bullet$ etc.

Morphisms are of the form eg.



$: \bullet^{\otimes 9} \longrightarrow \bullet^{\otimes 9}$

Example The generic Rumer-Teller-Weyl cat TL

$$TL = \langle \cdot, |, \cap, \cup \mid \wr = | = \wr \rangle$$

Morphisms:

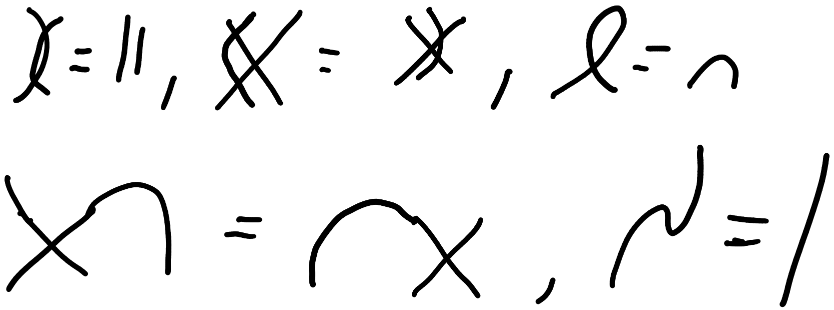
Fact: $TL \simeq 1Cob^{em}$

embedded in $\mathbb{R} \times [0,1]$

algebraic model

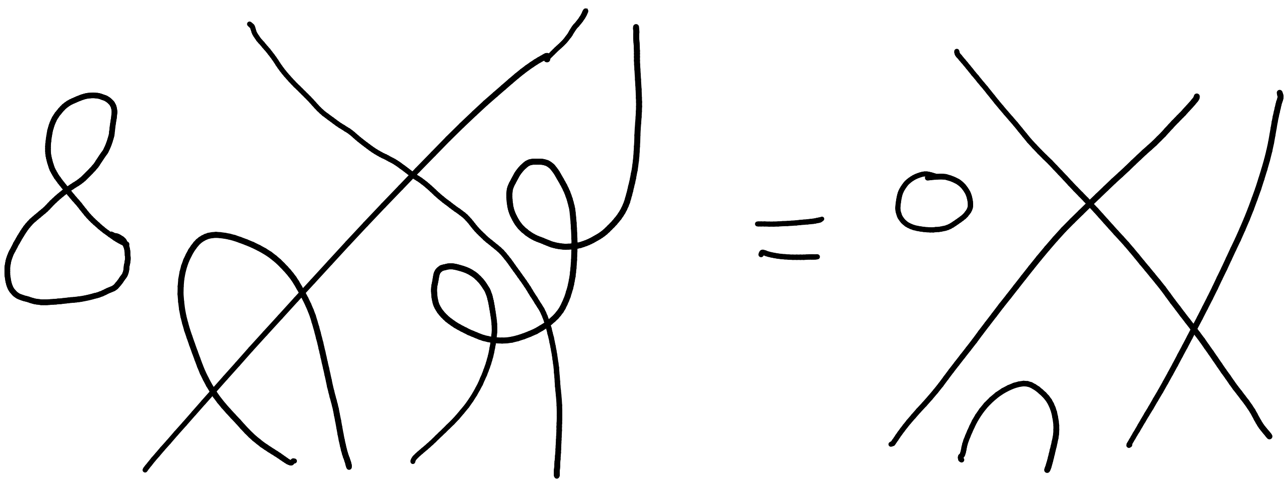
Example The generic Brauer cat Br

$\mathrm{Br} = \langle \bullet, \; X, \; \cap, \; \cup \mid$



+ unions

Morphisms:



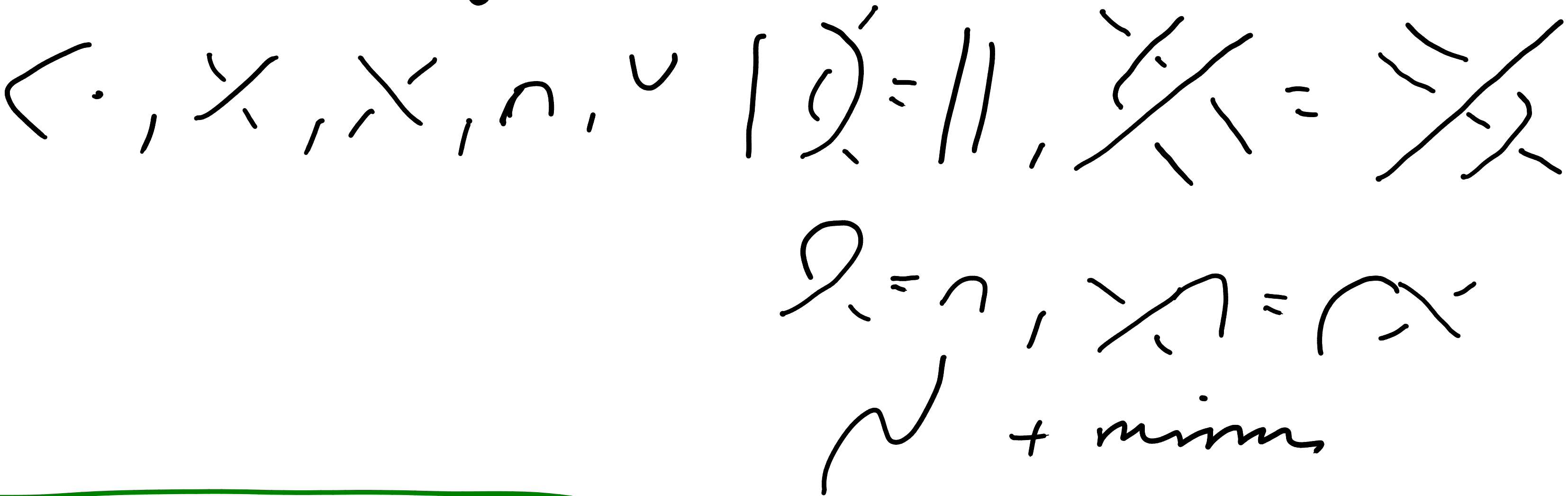
Fact:

$\mathrm{Br} \simeq 1\mathrm{Cob}$

algebraic model

Example The generic q Brauer cat $q\mathrm{Br}$

$q\mathrm{Br} =$



Fact: $q\mathrm{Br} \simeq 1\mathrm{Tan}$

$\uparrow$ algebraic model

Upshots of gen-rels:

- One can give algebraic definitions of cats appearing in the wild, eg. $\mathcal{B}r \simeq 1\mathcal{C}ob$

- If $\mathcal{C} = \langle X, w \mid R \rangle$, then one can define a functor $F: \mathcal{C} \to \mathcal{D}$ on X, w only and the only thing to check is whether R is preserved by F

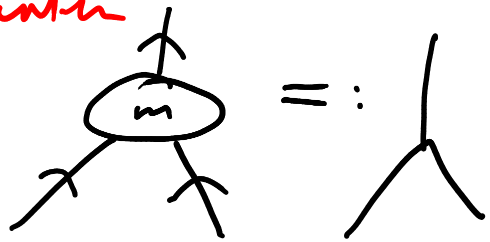
Example $F: \mathcal{B}r \to 1\mathcal{C}ob$ is easy to define, $G: 1\mathcal{C}ob \to \mathcal{B}r$ is hard to define

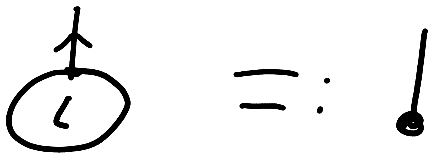
What else can we generalize? Algebras!

Def 3.3 An **algebra** $A \in \mathcal{C}$ is an object together with two structure morphisms

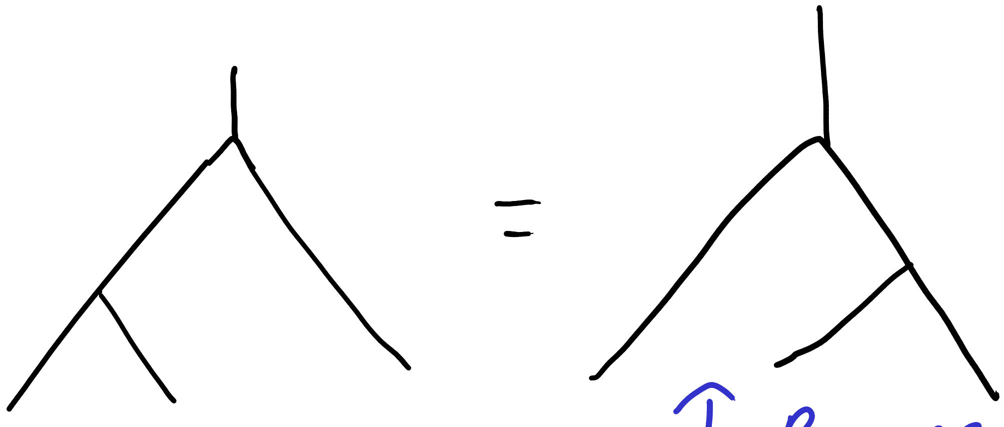
$$m : AA \longrightarrow A \qquad\qquad \iota : \mathbb{1} \longrightarrow A$$

multiplication $\qquad\qquad$ unit



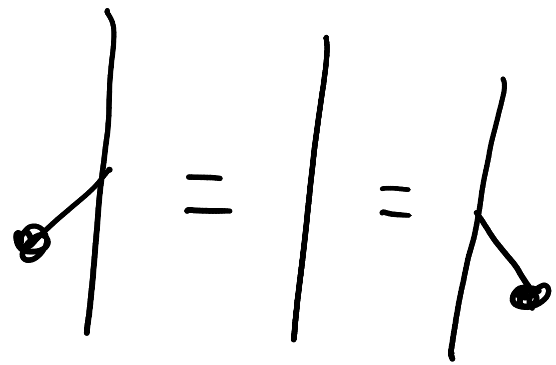


such that



asso

↑ Recall: Implies "real" asso



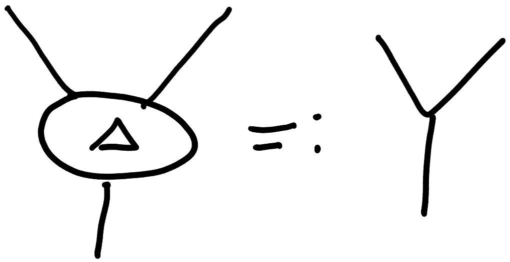
unitality

Similarly, coalgebras:

Def 3.4 A coalgebra $C \in \mathcal{C}$ is an object together with two structure morphisms

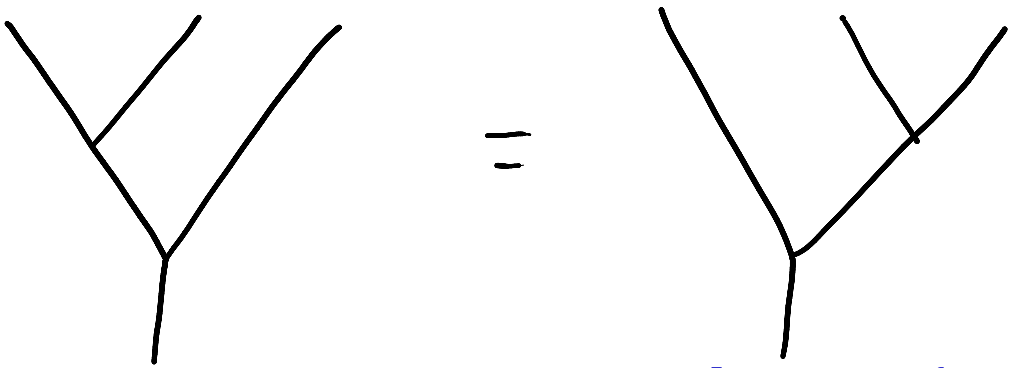
$$\Delta : A \longrightarrow AA \qquad\qquad \varepsilon : A \longrightarrow \mathbb{1}$$

Comultiplication

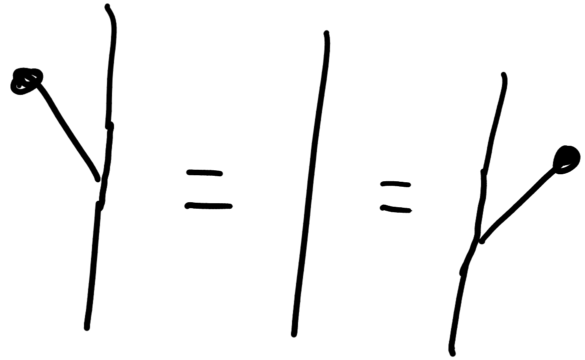


counit

such that



coasso



counitality

↑ This implies "real" coasso

Examples – Algebras in $(\mathrm{Set}, \times)$ are monoids

– Algebras in $(\mathrm{Vect}_{\mathbb{K}}, \otimes_{\mathbb{K}})$ are algebras

– Same for coalgebras

Formally one should go to the strictification

– $\mathbb{1}$ is always a (co)algebra

– For any submonoid $N \subset M$, $N \in \mathrm{Vect}(M)$ is a (co)algebra object

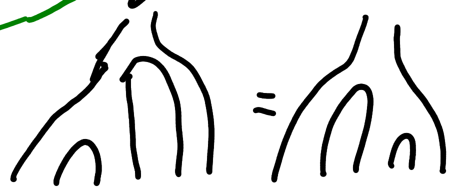
(Think: Subgroups of groups give subalgebras of $\mathbb{K}[G]$)

Same for twists

– For TL, Br, qBr $\quad A = \bullet\bullet$ is a (co)algebra

$m: \bullet\bullet\bullet\bullet \to \bullet\bullet$, $\iota = \cup$

e.g. asso:



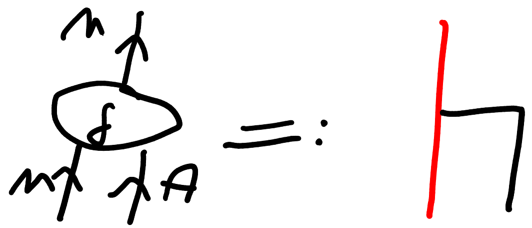
Next: Generalizing actions

Def 3.5 (The comodule version works verbatim, similarly for left $\leftrightarrow$ right)

A right module object $M \in \mathcal{C}$ of an algebra $A \in \mathcal{C}$ is an object of $\mathcal{C}$ together with a structure morphism

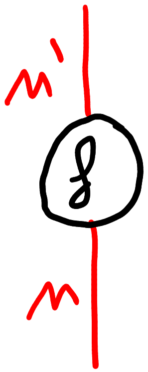
$f : MA \longrightarrow M$

right action

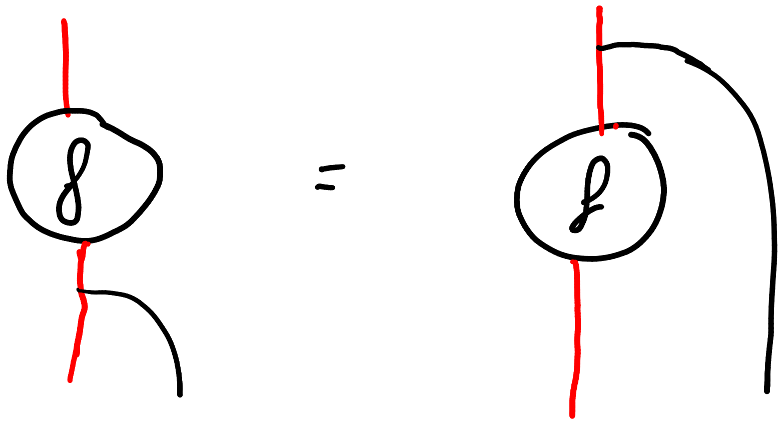


such that

A module homomorphism is the evident notion:



satisfying



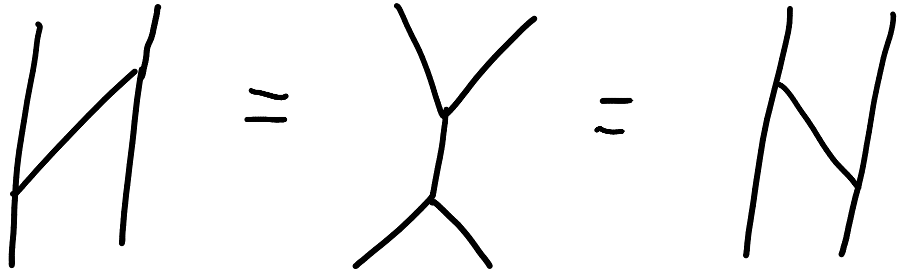
$\rightsquigarrow$ Get a category $\mathrm{Mod}_{\mathcal{C}}(A)$

Example

All of these generalize the classical notions of algebras, modules etc. if we work in $\mathrm{Vect}_{\mathbb{K}}$.

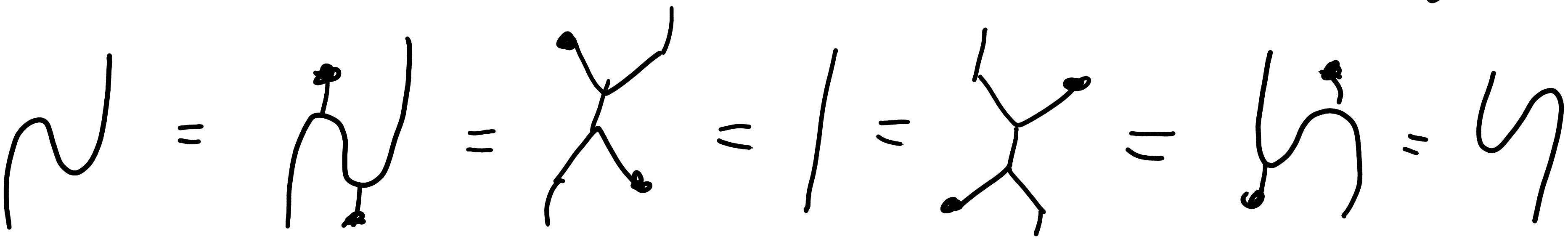
Next, an instance why graphical calculus is cool

Def 3.6 Let $F \in \mathcal{C}$ be an algebra and a coalgebra. F is called Frobenius if



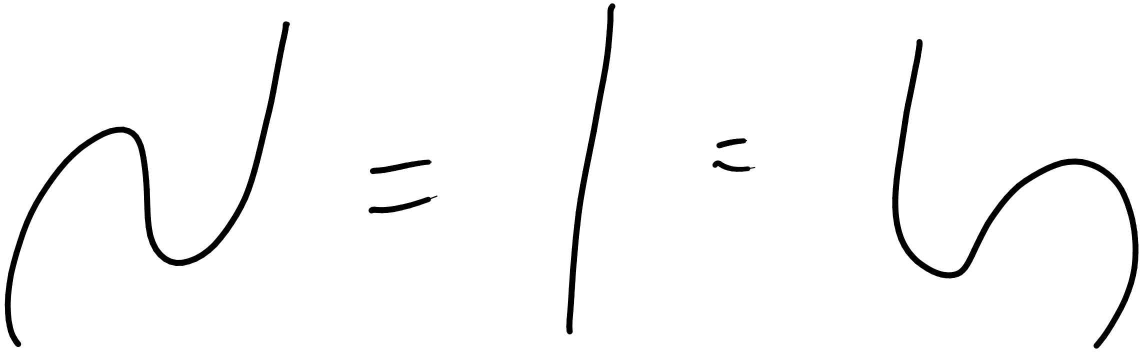
For such F we can define (co)pairings

satisfying

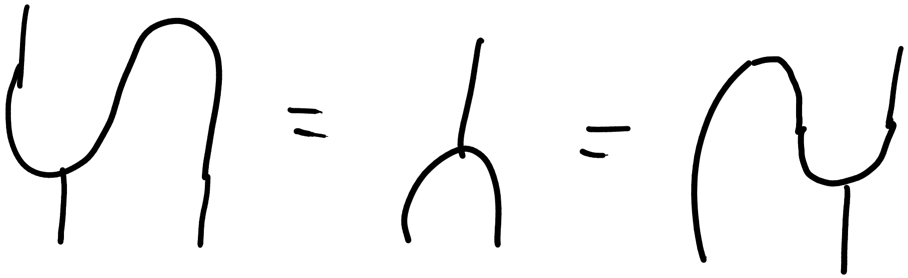


Easy fact:

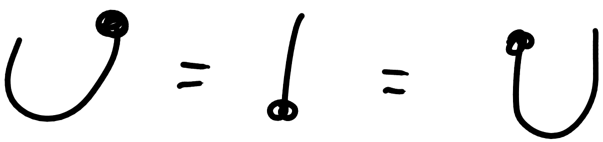
The following hold for such $\mathcal{F}$.



(see above)



+ mirror



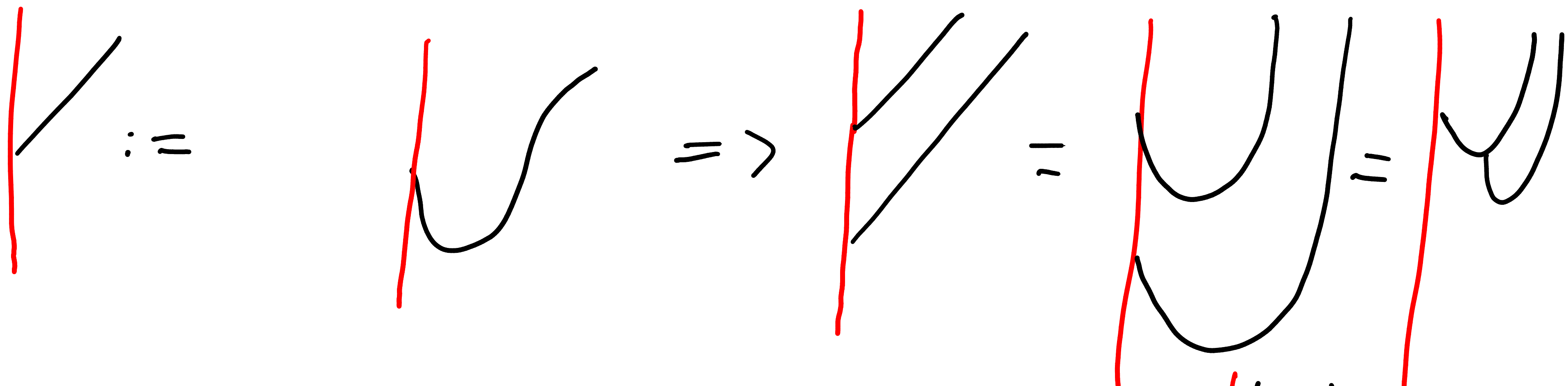
+ mirror

$\Rightarrow$ Planar isotopies

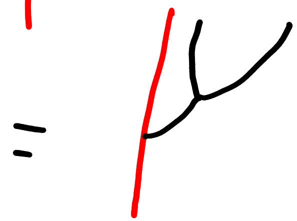
Proposition 3.7 (Quite a mess if you want to do it in $\mathrm{Vect}_{\mathbb{K}}$ classically)

Every right A module M has a compatible right A comodule structure and vice versa

Proof: From module $\Rightarrow$ comodule



The other direction works verbatim.



etc.