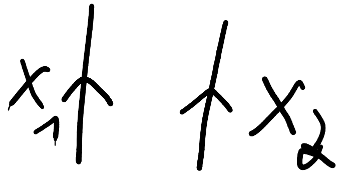
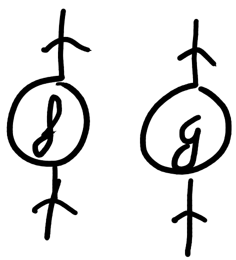
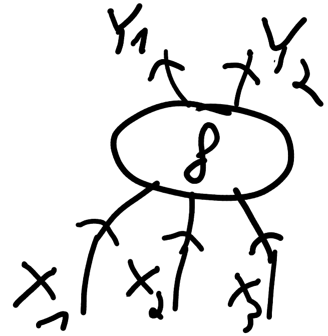
Lecture 2 Monoidal cats I - Defs, Examples + graphics

What are the correct axioms to get a planar graphical calculus?

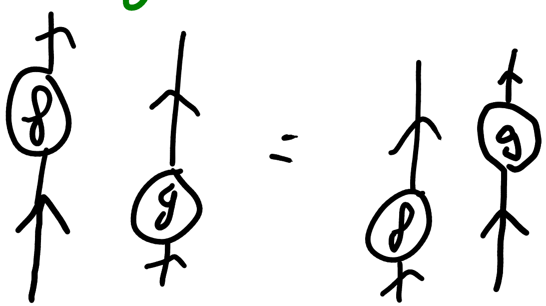
$X_1 \otimes X_2 \rightsquigarrow$ 

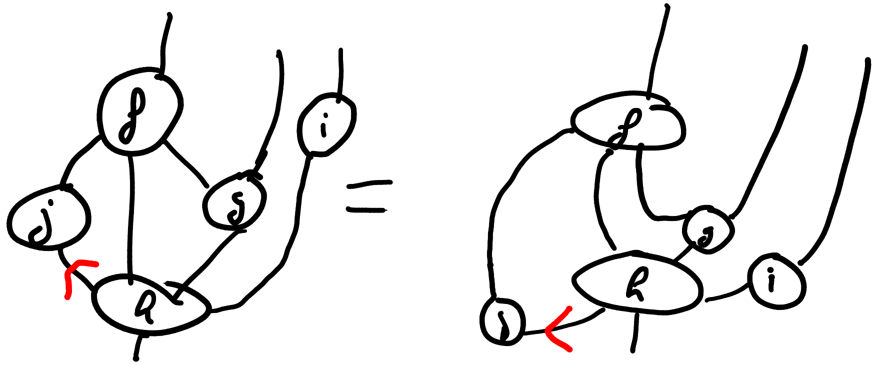
$f : X_1 \otimes X_2 \otimes X_3 \longrightarrow Y_1 \otimes Y_2$

$f \otimes g \rightsquigarrow$ 



Respecting ↑:

Ok 

Bad 

$\mathcal{C} \times \mathcal{C} \rightsquigarrow$ cat with objects (X, Y) morph. (f, g)

Motivating example

$\otimes : \mathrm{Set} \times \mathrm{Set} \longrightarrow \mathrm{Set}$

$$(X, Y) \longmapsto X \times Y =: X \otimes Y$$

Note: - $\{\bullet\} = \mathbb{1}$ is a unit object i.e. $\mathbb{1} \otimes X \underset{\neq}{\simeq} X \underset{\neq}{\simeq} X \otimes \mathbb{1}$

- This is only weakly associative, i.e.

$$(X \otimes Y) \otimes Z \underset{\neq}{\simeq} X \otimes (Y \otimes Z)$$

$$((x, y), z) \qquad (x, (y, z))$$

- Also the unit is only weak $(\bullet, x) \neq x \neq (x, \bullet)$

Def 2.1 A monoidal cat $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$
$\otimes$-cat
consist of

- a category $\mathcal{C}$
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$id_X \otimes id_Y = id_{X \otimes Y}$
$(k \otimes h) \circ (g \otimes f) = (k \circ g) \otimes (h \circ f)$

- a unit $\mathbb{1}$
- natural iso $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$

called associator

- natural iso $l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X$

called unitors $r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X$

Recall: naturally means that a $\Box$-commutes

such that $\bigcirc$-equation and the Δ-equation hold. (I.e. commute)

$\triangle$

"Unit"

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes id"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "id \otimes \ell_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

$\square$

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, "(f \otimes g) \otimes h"] \arrow[d, "\alpha_{U,V,W}"] & (X \otimes Y) \otimes Z \arrow[d, "\alpha_{X,Y,Z}"] \\
U \otimes (V \otimes W) \arrow[r, "f \otimes (g \otimes h)"'] & X \otimes (Y \otimes Z)
\end{tikzcd}$$

From now on

$XY := X \otimes Y$

$\pentagon$

"Asso"

$$\begin{tikzcd}
& ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\alpha_{W \otimes X,Y,Z}"] \arrow[dr, "\alpha_{W,X,Y} \otimes id"] & \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W,X,Y \otimes Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\alpha_{W,X \otimes Y,Z}"] \\
W \otimes (X \otimes (Y \otimes Z)) & & W \otimes ((X \otimes Y) \otimes Z) \arrow[ll, "id \otimes \alpha_{X,Y,Z}"]
\end{tikzcd}$$

Why is this a bad Definition?

Wrong def / monoid	Correct def / monoid
- A set M	- Same
- multiplication $\cdot : M \times M \to M$	- Same
- unit $1 \in M$	- Same
- asso $(k \cdot m) \cdot n = k \cdot (m \cdot n)$	- asso : all bracketings are the same

Note: $(k \cdot m) \cdot n = k \cdot (m \cdot n)$ is not associativity, but it is so well-known that

$$\text{wrong} \Rightarrow \text{Correct}$$

that this is usually ignored...

Theorem 2.2 (Coherence)

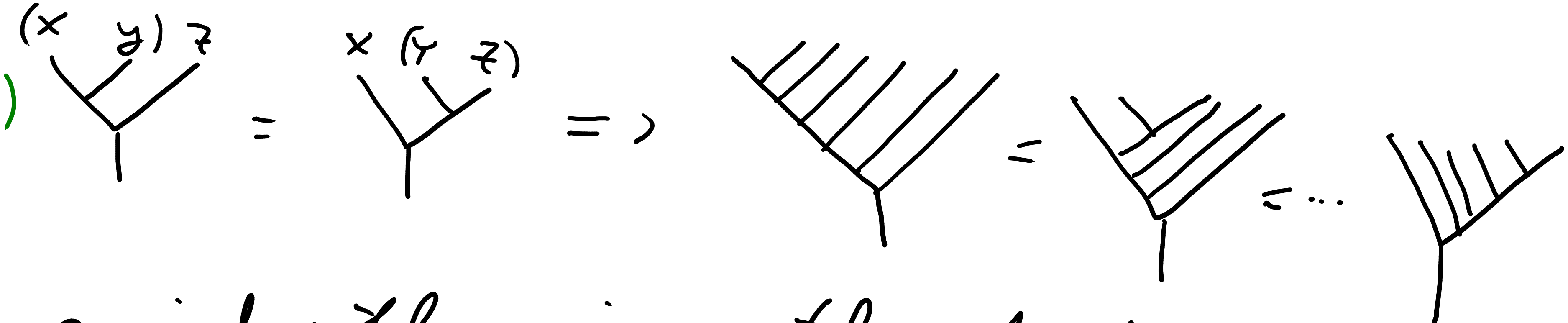
All ways to build associators and unitors are the same

Proof (sketch)

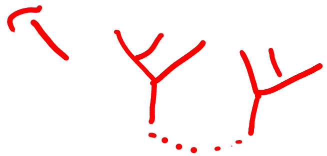
"monoids"

Let analyze the 1-dim case:

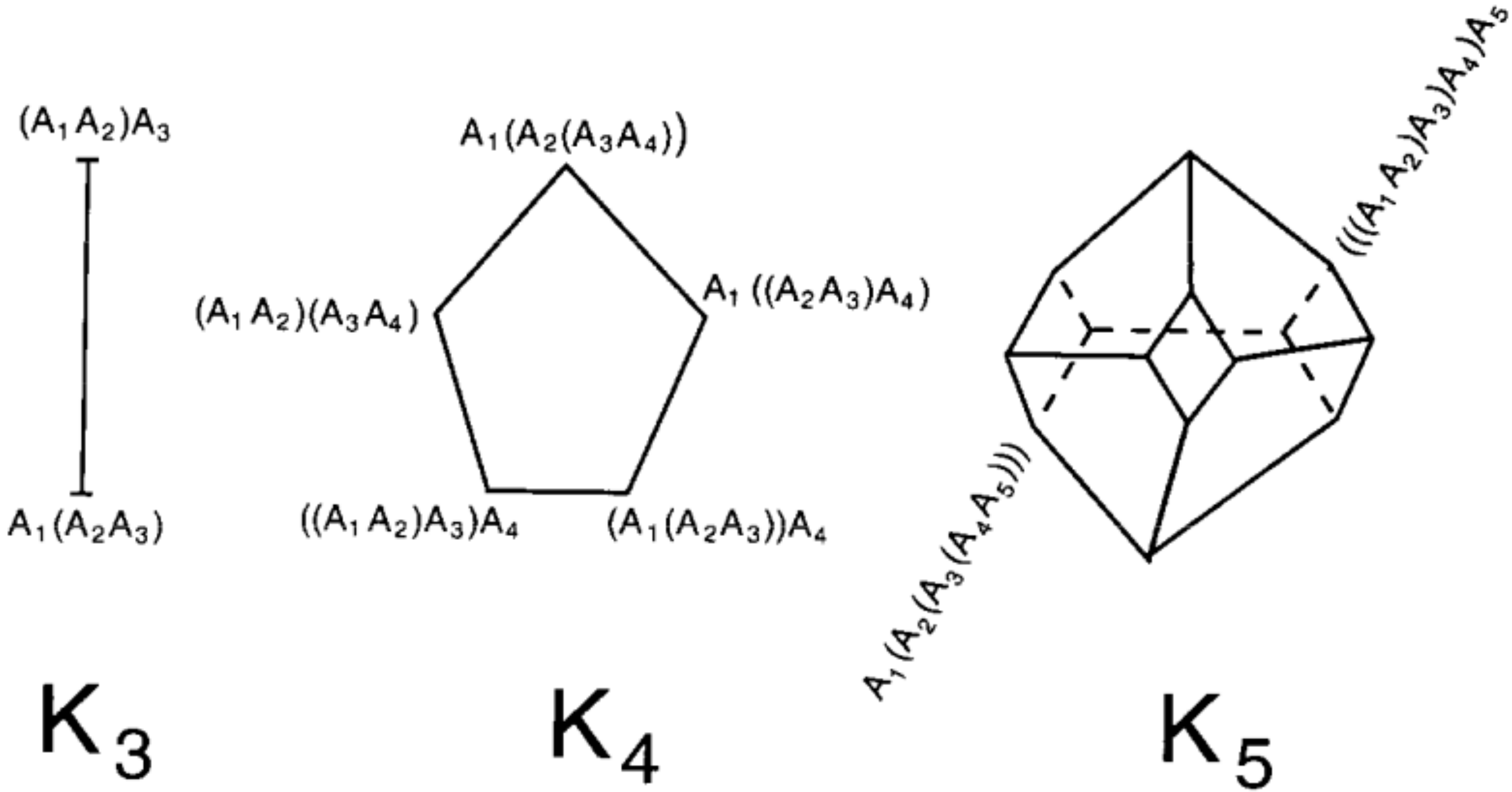
(*)



Or said otherwise, the 1-dim CW complex K_n "graph" which one gets by adding an edge for each (*) is connected i.e. $\pi_0(K_n) = 1$



Play the same game:



Then $\pi_1(K_n) = 1$ implies coherence. Similarly for unitors.

<u>Def 2.3</u> $\mathcal{C}$ is called strict if α, ℓ, r are all identities

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Examples – Set with $\otimes = \times$, $\mathbb{1} = \{\cdot\}$ non-strict

– Set with $\otimes = +$ (dis. union), $\mathbb{1} = \emptyset$ non-strict

– $\mathrm{Vect}_{\mathbb{K}}$ with $\otimes = \otimes_{\mathbb{K}}$, $\mathbb{1} = \mathbb{K}$ non-strict

– $\mathrm{Vect}_{\mathbb{K}}$ with $\otimes = \oplus$, $\mathbb{1} = \{0\}$ non-strict

(Note that their skeletons are strict)

– 1Cob, 1Tan, 1State strict

$\otimes$ = juxtaposition, $\mathbb{1} = \emptyset$

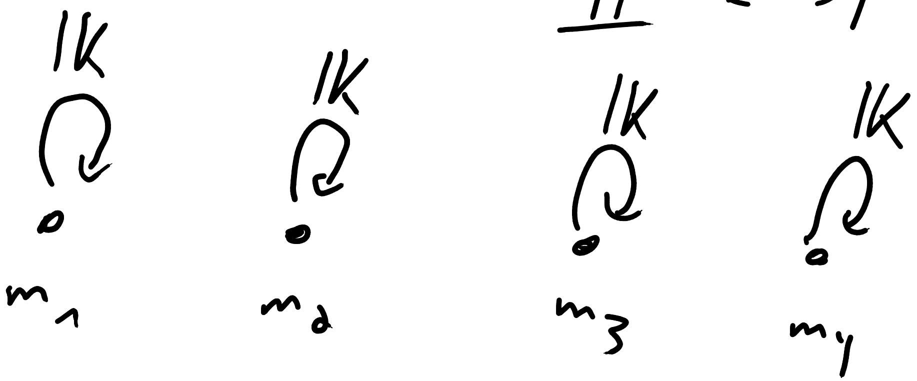
- Groups / Monoids as $\otimes$ - cats

$\mathrm{Vect}(M)$ $\quad$ Ob = elements of M

Mor = only $\mathbb{K}\cdot id$

$m \otimes n = m\cdot n$

$\underline{\mathbb{1}} = 1$



strict

3 cocycle
$\omega : M \times M \times M \to \mathbb{K}^*$
satisfying the
pentagon - axiom

- $\mathrm{Vect}^{\omega}(M)$ same but

α = 3 cocycle

$l = \alpha(1,1,m)^{-1}$

$r = \alpha(m,1,1)$

Example: $M = \mathbb{Z}/2 = \langle 1, \sigma \rangle$

$\alpha = 1$ except $\alpha(\sigma,\sigma,\sigma) = -1$

non-strict $\omega \neq 1$, and skeletal

Question Can we ignore α, l, r, i.e. is every $\mathcal{C}$ "strictifyable"?

The last example shows that one can not simply go to the skeleton.

However, the answer is "Yes" as we will see.

In particular, we will usually ignore α, l, r and work with strict $\otimes$-cats.

First things first:

Def 2.4 A ⊗-functor $F = (F, \xi_1, \xi) : \mathcal{C} \to \mathcal{C}'$ consist of:

- a functor F
- natural isomorphisms
$$\xi_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y)$$
- natural isomorphism $\xi_1 : \mathbb{1}' \xrightarrow{\sim} F(\mathbb{1})$

such that one has commuting diagrams

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "\xi\otimes\mathrm{id}"] \arrow[d, "\alpha'"'] & F(X \otimes Y) \otimes' F(Z) \arrow[r, "\xi"] & F((X \otimes Y) \otimes Z)) \arrow[d, "F(\alpha)"] \\
F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[r, "\mathrm{id}\otimes\xi"] & F(X) \otimes' F(Y \otimes Z) \arrow[r, "\xi"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

⬡ - equation

□ - equation

$$\begin{tikzcd}
I' \otimes F(X) \arrow[r, "\xi_0\otimes\mathrm{id}"] \arrow[d, "\lambda'"'] & F(I) \otimes' F(X) \arrow[d, "\xi"] \\
F(X) & F(I \otimes X) \arrow[l, "F(\lambda)"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(X) \otimes I' \arrow[r, "\mathrm{id}\otimes\xi_0"] \arrow[d, "\rho'"'] & F(X) \otimes' F(I) \arrow[d, "\xi"] \\
F(X) & F(X \otimes I) \arrow[l, "F(\rho)"']
\end{tikzcd}$$

Def 2.5 A $\otimes$-nat trafo $\varphi : F \Rightarrow G$ is a nat trafo such that

$$\varphi_{\mathbb{1}} \xi_0^F = \xi_0^G \quad , \quad \varphi_{XY} \xi_{XY}^F = \xi_{XY}^G \varphi_X \varphi_Y$$

$\mathcal{C} \simeq \mathcal{D}$ as $\otimes$-cats if $\exists F : \mathcal{C} \to \mathcal{D}$ which is an equivalence and a $\otimes$-functor

Examples
- $\underset{\times}{\mathrm{Set}} \simeq \mathrm{Sh}(\underset{\times}{\mathrm{Set}})$ as $\otimes$-cats
- $\underset{\otimes_{\mathbb{K}}}{\mathrm{Vect}_{\mathbb{K}}} \simeq \underset{\otimes_{\mathbb{K}}}{\mathrm{Mat}_{\mathbb{K}}}$ as $\otimes$-cats
- $\mathrm{Vect}\,M \not\simeq \mathrm{Vect}^{\omega} M$ as $\otimes$-cats in general, but always as cats.

Similarly define $\otimes$-subcats etc.

Lemma 2.6

Composition of $\otimes$-functors are $\otimes$-functors

Composition of $\otimes$-nat trafos are $\otimes$-nat trafos

Theorem 2.7

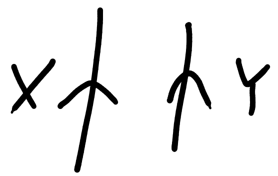
Every $\otimes$-cat is $\otimes$-equi to a strict one.

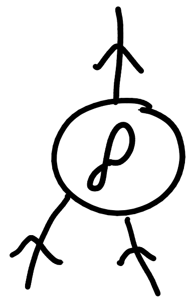
No α, ℓ, r :D

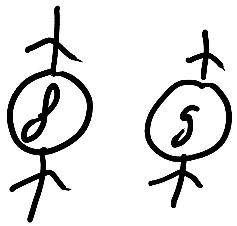
Proof, sketch

Use some $\otimes$-version of Yoneda.

Since we can wlg assume that $\mathcal{C}$ is strict, we can use graphical calculus!

$XY \rightsquigarrow$ 

$f: XY \to Z$  etc.

$fg \rightsquigarrow$ 

Rules: Planar isotopies respecting $\uparrow$. (*)

Theorem 2.8

The graphical calculus is consistent, i.e. two morphisms are the same iff their graphical repre. are related by a (*) isotopy.

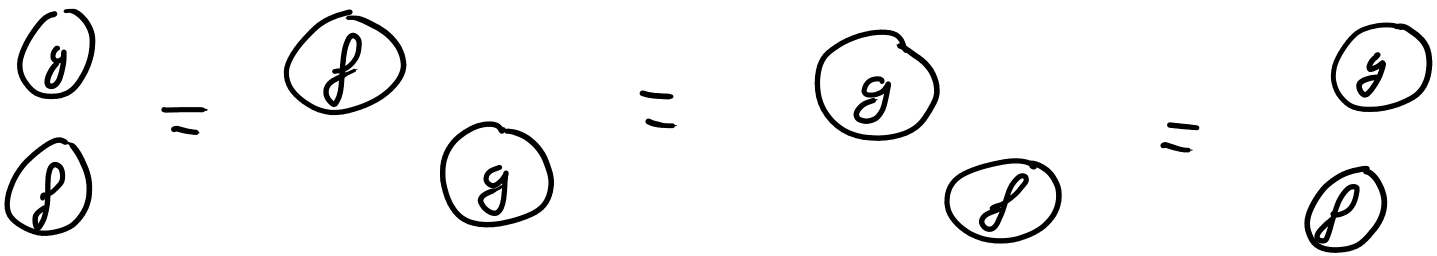
To wrap up, let me do some proofs using graphical calculus.

First, note that $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ is a monoid.

Proposition 2.9

$\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is commutative

Proof

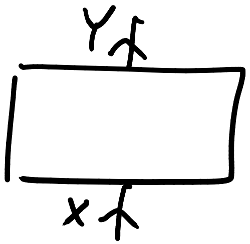


Proposition 2.10

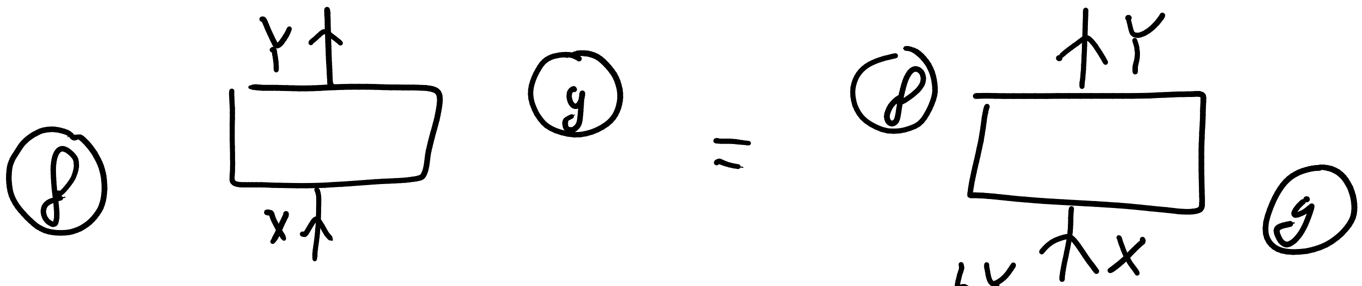
We have two commuting actions

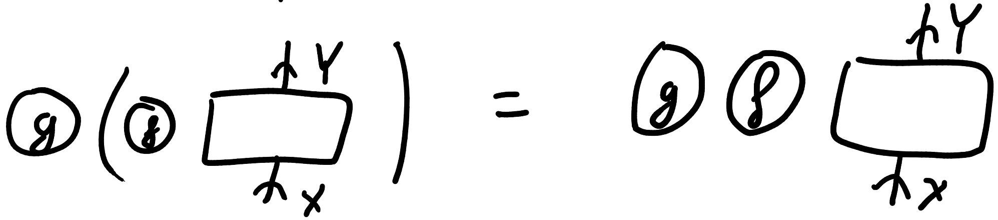
$$\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \curvearrowright \mathrm{Hom}_{\mathcal{C}}(X, Y) \curvearrowleft \mathrm{End}_{\mathcal{C}}(\mathbb{1})$$

Thus, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ bimodule.

Proof: 

= some morphism in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$





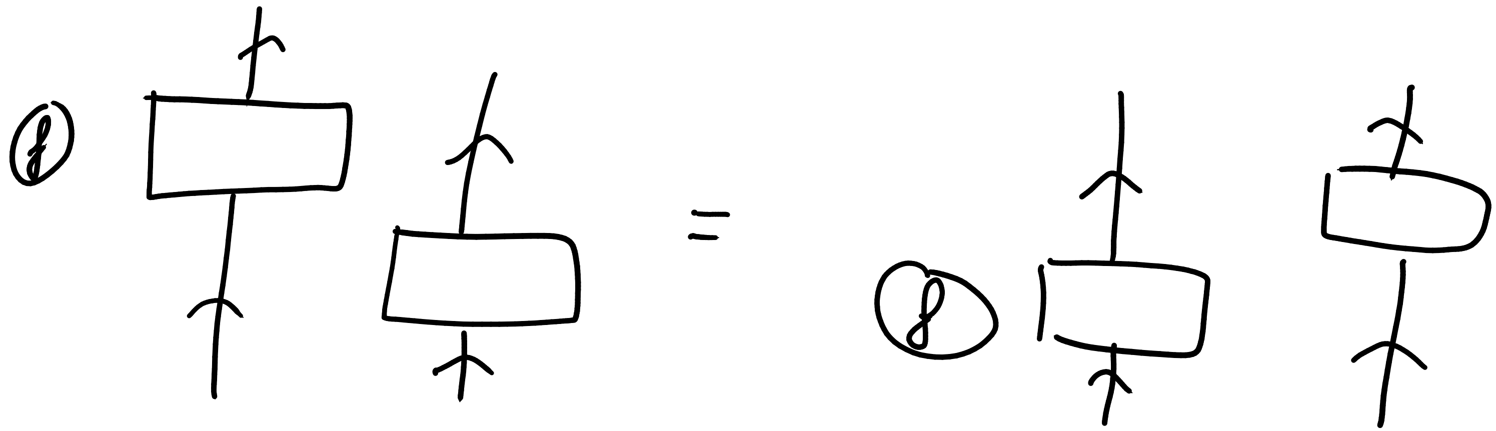
+ mirror

Proposition 2.11

The actions of $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are compatible with $\circ$ and $\otimes$.

Proof

$\otimes$



$\circ$