Lecture 1 Categories – Definitions, Examples + Strings

"Classical mathematics is based on sets.
Modern mathematics is based on categories"

The Rosetta Stone

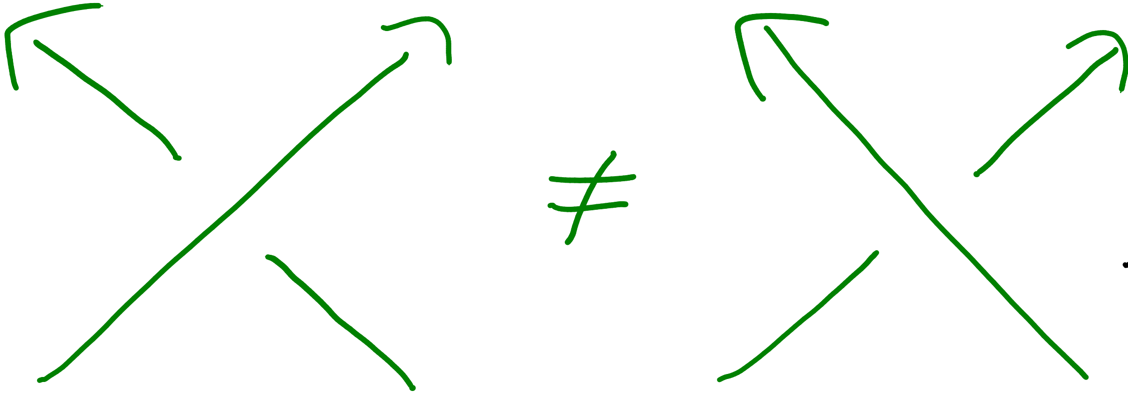
Cat. theory	Algebra	Topology	Physics	Logic
objects X	alg. data X	manifold X	system X	proposition X
morphism $f: X \longrightarrow Y$	relation $f: X \longrightarrow Y$	cobordism $f: X \longrightarrow Y$	process $f: X \longrightarrow Y$	proof $f: X \longrightarrow Y$
monoidal product $X \otimes Y$	product data $X \otimes Y$	disjoint union $X \otimes Y$	joint system $X \otimes Y$	conjunction $X \otimes Y$
monoidal product $f \otimes g$	parallel relations $f \otimes g$	dis. union $X \otimes Y$	parallel process $f \otimes g$	parallel proofs

Goal of this lecture

Explain how category theory gives a way to study algebra + topology (and baby versions of physics and logic)

Non-commutative alg. structures

Particles in $\mathbb{R}^3$



Links/Knots

Ribbon logic

I will explain the categorical analog

We begin at the beginning:

Def 1.1 A **category** $\mathcal{C}$ consists of

- A collection of **objects** $Ob(\mathcal{C})$, $X \in \mathcal{C}$
- Hom spaces, i.e. sets $Hom_{\mathcal{C}}(X, Y)$ of **morphisms** $f: X \to Y$, $f \in \mathcal{C}$

Such that:

- Composition morphisms $gf: X \to Z$ for $f: X \to Y$, $g: Y \to Z$
- Identities $id_X: X \to X$, $id_Y f = f = f id_X$
- Associativity $h(gf) = (hg)f$

Examples

- Categories generalize monoids M.
M is a cat. with one (completely unimportant) object $\bullet$ and $\mathrm{Hom}_M(\bullet, \bullet) = M$

- Categories generalize sets. Set = Cat of sets, $\mathrm{Ob}(\mathrm{Set})$ = sets, morphisms are functions

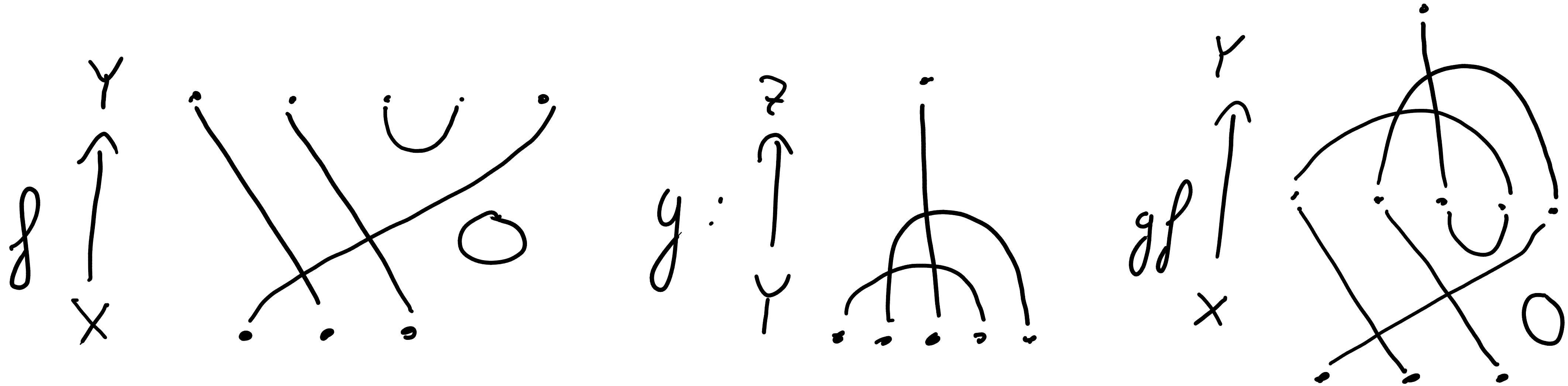
- Categories generalize vector spaces.
$\mathrm{Vect}_{\mathbb{K}}$ = Cat of $\mathbb{K}$-VS, $\mathrm{Ob}(\mathrm{Vect}_{\mathbb{K}}) = \mathbb{K}$-VS, morphisms = linear maps

Cats are traditionally named after their objects, but morphisms are the main players

Examples – 1Cob Cat of 1-dim cobordisms

Ob = 0-dim manifolds aka points

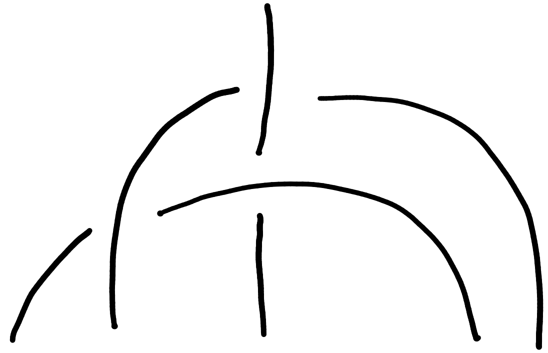
Mor = cobordisms aka lines



– 1Tan Same as 1Cob, but remembering the embeddings of the lines in $\mathbb{R}^3$

f :

g :



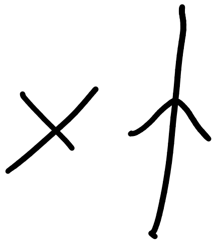
– 1State objects collection of particles

morphisms worldlines, i.e. how particles move in $\mathbb{R}^3$

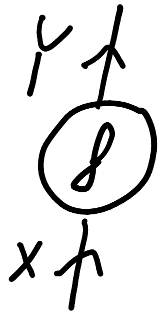
Sometimes also called 1orTan "oriented tangles"

Feynman diagrams for categories

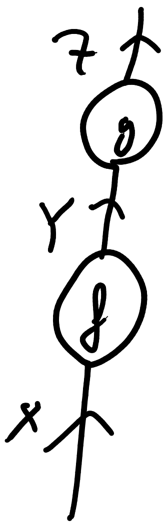
Objects



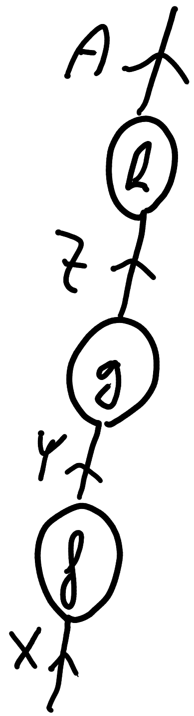
Mor



Composition



Assoc is implicit:



A map between categories is:

Def 1.2 A functor $F : \mathcal{C} \to \mathcal{D}$ is a map sending

- $X \in \mathcal{C}$ to $F(X) \in \mathcal{D}$ "Ob $\mapsto$ Ob"
- $f : X \to Y$ to $F(f) : F(X) \to F(Y)$ "Mor $\mapsto$ Mor"

such that

- "Identities are preserved" $F(id_X) = id_{F(X)}$
- "Composition is preserved" $F(gf) = F(g)F(f)$

Examples

- Functors generalize monoid maps
  $F: M \to M'$ is given by $F(\bullet) = \bullet$ and $F(gf) = F(g)F(f)$
- Functors generalize models
  $F: M \to Set$, $F(\bullet) =$ some set , $F(f) =$ some map
- Functors generalize representations
  $F: M \to Vect_{\mathbb{K}}$, $F(\bullet) =$ some $\mathbb{K}$-VS, $F(f) =$ some linear map
- Functors generalize forgetting, e.g.
  $F: Vect_{\mathbb{K}} \to Set$, $F(VS) =$ underlying set
  $F($linear map$) =$ underlying map

- Functors generalize freeness

$F: Set \longrightarrow Vect_{\mathbb{K}}$, $F(set)$ = free $\mathbb{K}$-VS on this set

$F(f: X \to Y)$ = linearization of f

Functors give us another example of a category:

Cat $\quad Ob(Cat)$ = categories

$Mor(Cat)$ = functors

A map between functors is called nat. trafo:

Def 1.3 $F, G : \mathcal{C} \to \mathcal{D}$. A natural transformation $\varphi : F \Rightarrow G$ is a collection

$$\{\varphi_X : F(X) \to G(X)\}_{X \in Ob(\mathcal{C})}$$

such that

$$F \; \Downarrow \varphi \; G \qquad \begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\varphi_X"'] & F(Y) \arrow[d, "\varphi_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

commutes

Example Nat trafos generalize intertwiners ("maps of representations")

$F : M \longrightarrow Vect_{\mathbb{K}}$ $\qquad$ $G : M \longrightarrow Vect_{\mathbb{K}}$

$F(\bullet) = X \leftarrow$ some VS $\qquad$ $G(\bullet) = Y \leftarrow$ some other VS

$\varphi : F \Rightarrow G$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_x"'] & X \arrow[d, "\varphi_x"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

$f \varphi_x = \varphi_x g$

Some notions which we will need:

- $f: X \to Y$ is called an isomorphism if $\exists$ $g: Y \to X$ such that $gf = id_X$, $fg = id_Y$

Usual Yoga: Inverses are unique if they exist. Write f^{-1}

- $X \simeq Y$ (isomorphic) if $\exists\, f: X \to Y$ isomorphism

$\rightsquigarrow$ gives an equi. rel. on $Ob(\mathcal{C})$, so $Ob(\mathcal{C})/_{\simeq}$ makes sense

$\rightsquigarrow$ new category $Sk(\mathcal{C})$ = Skeleton, $Ob = Ob(\mathcal{C})/_{\simeq}$

Example

Two $\mathbb{K}$-VS are isomorphic iff (this means if and only if) they have the same dimension

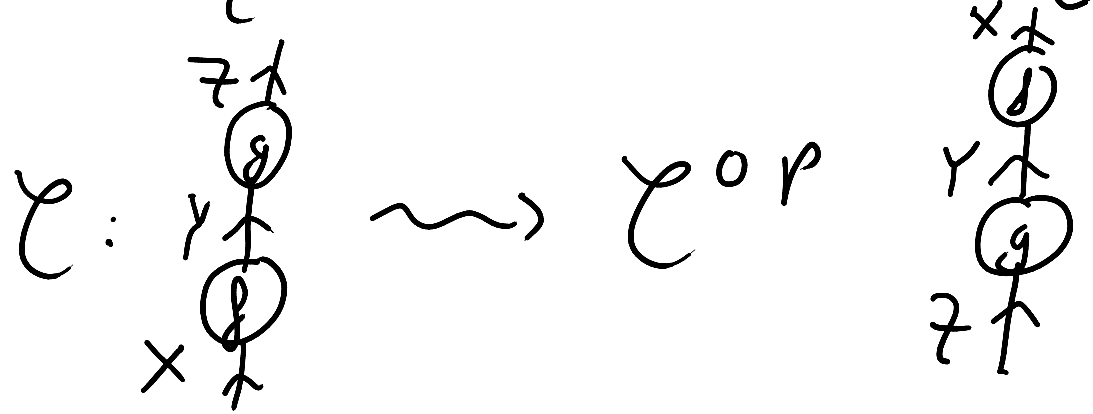
$\Rightarrow$ $\operatorname{Sk}(\mathrm{Vect}_{\mathbb{K}}) = \mathrm{Mat}_{\mathbb{K}}$ "Matrices"

$\mathrm{Ob}(\mathrm{Mat}_{\mathbb{K}}) = \{\mathbb{K}^{\alpha}\}$

Mor $f: \mathbb{K}^{\alpha} \longrightarrow \mathbb{K}^{\beta}$ "honest matrices"

- The **opposite** category $\mathcal{C}^{op}$, $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$

$\mathrm{Hom}_{\mathcal{C}^{op}}(Y, X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ "reversed $\circ$"



- **Subcategory** $\mathcal{C} \subset \mathcal{D}$, $Ob(\mathcal{C}) \subset Ob(\mathcal{D})$

$\mathrm{Hom}_{\mathcal{C}}(X, Y) \subset \mathrm{Hom}_{\mathcal{D}}(X, Y)$ + ids are in $\mathcal{C}$

$\mathcal{C}$ is **full** if $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(X, Y)$

Example

$\mathcal{C} = \mathrm{fdVect}_{\mathbb{k}} \subset \mathrm{Vect}_{\mathbb{k}} = \mathcal{D}$ is a full subcat

Crucial Example

Cat of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

Ob = Functors $F : \mathcal{C} \longrightarrow \mathcal{D}$

Mor = nat trafos $\varphi : F \Rightarrow G$

Example $\mathrm{Fun}(M, \mathrm{Vect}_{\mathbb{K}})$ are all representations of the monoid M

Note: Via $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ we get the notion of functors $F \cong G$ being isomorphic

What should $\mathcal{C} \simeq \mathcal{D}$ be?

First try

$\mathcal{C}$ and $\mathcal{D}$ are **isomorphic** if $\exists\, F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G = F^{-1}$ meaning $GF = id_{\mathcal{C}}$ $\quad FG = id_{\mathcal{D}}$

This is a really $\boxed{\text{bad}}$ notion in category theory. Why? Because isomorphic cats have the same number of objects, but we do not care about objects.

Def 1.4 A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ has a quasi-inverse $G : \mathcal{D} \longrightarrow \mathcal{C}$ if

$$GF \simeq id_{\mathcal{C}}, \quad FG \simeq id_{\mathcal{D}}$$

F is called an equivalence if it has a quasi-inverse.

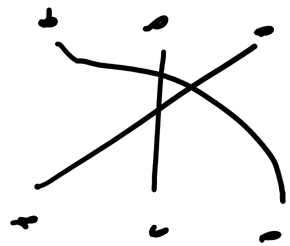
$\mathcal{C} \simeq \mathcal{D}$ (equivalent) if $\exists\, F : \mathcal{C} \to \mathcal{D}$ an equivalence

Examples

- $\mathcal{C} \simeq \mathrm{Sk}(\mathcal{C})$
- In particular, $\mathrm{Vect}_{\mathbb{K}} \simeq \mathrm{Mat}_{\mathbb{K}}$
- Modern algebraic geometry in one line

  $(\text{affine varieties}/\mathbb{K})^{op} \simeq (\text{fin. gen. red. } \mathbb{K} \text{ algebras})$
- $\mathrm{finSet}^{bij} \simeq \mathrm{Sym}$

  $\mathrm{Ob} = \mathbb{N}$

  $\mathrm{Mor} =$ 

only bijections

"The sym. groups"

Proposition 1.5

$F : \mathcal{C} \to \mathcal{D}$ is an equivalence iff:

- F is dense, i.e. $\forall Y \in \mathcal{D}\ \exists X \in \mathcal{C} : F(X) \simeq Y$
- F is faithful, i.e.

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$

is injective

- F is full, i.e.

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$

is surjective.

Proposition 1.6

We have the Yoneda embedding, i.e.

$$\mathrm{Hom}(-,-) : \mathcal{C}^{op} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$$

faithful also $\mathcal{C}$ is a subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$

$$X \longmapsto \mathrm{Hom}(X, X)$$

$$f : X \to Y \longmapsto \mathrm{Hom}(X, X) \xrightarrow{f \circ -} \mathrm{Hom}(Y, Y)$$

$$g \longmapsto f \circ g$$