HARMONIC ANALYSIS
2022/23: SPRING SEMESTER

DIOGO OLIVEIRA E SILVA

1. Webpage. Updated information about the course can be found on Fénix.

2. Lectures. The lectures will take place on Tuesdays 9-11AM and Thursdays 3-5PM in P3 (Pavilhão de Matemática). The first lecture will be held on Thursday, February 23rd.

3. Overview. This is a fast-paced course intended for Master and PhD students which surveys a wide array of results in classical and modern euclidean harmonic analysis. Emphasis will be given to the real-variable aspects of the subject over those more closely linked to complex analytic and harmonic functions. The course divides into three sections:

I. Classical operators in Euclidean space
II. Sharp inequalities in harmonic analysis
III. Fourier restriction theory

In part I, we present a sample of classical results that can be loosely grouped into three categories: maximal averages, singular integrals, and oscillatory integrals. In part II, we discuss some inequalities whose derivation is not too difficult if optimal constants are not demanded. The fact that these inequalities do not follow from simple convexity turns the determination of the corresponding sharp constants and the cases of equality into formidable problems, which we will address insofar as the theory is known. Finally, part III is meant as an introduction to the fascinating subject of Fourier restriction theory. Time permitting, I will try to highlight the interplay between restriction theory and some challenging research directions in modern harmonic analysis.

4. Topics. The following topics will tentatively be addressed:

- Fourier inversion and other basic stuff
- Convergence of Fourier series: elementary theory
- The Hardy–Littlewood maximal function
- Singular integral operators
- Sharp inequalities: Hardy–Littlewood–Sobolev and Hausdorff–Young
- Oscillatory integrals and stationary phase
- On curvature in higher-dimensional harmonic analysis: Fourier restriction theory; connections with (i) the Kakeya problem, (ii) Bochner–Riesz summation methods, (iii) decoupling theory for the Fourier transform.

Date: February 20th, 2023.

5. Prerequisites. Familiarity with the following concepts will be useful.

- *Measure theory:* Dominated convergence theorem; monotone convergence theorem; Fatou's lemma; theorems of Tonelli and Fubini; Riesz representation theorem; Radon–Nikodym theorem.
- *Point set topology:* Stone–Weierstraß theorem; Urysohn's lemma; topological vector spaces; weak and weak-$\star$ topologies.
- *Functional analysis:* Hilbert and Banach spaces; L^p-spaces: duality; Sobolev spaces; Banach–Alaoglu theorem; closed graph theorem; open mapping theorem; uniform boundedness principle.
- *Complex analysis:* Maximum modulus principle; residue theorem; identity theorem for analytic functions; mean value theorem for holomorphic functions.

6. Course credit and exam. There will be five problem sheets, each accounting for 10% of the final grade. Discussions are encouraged, but solutions should be individually typed, preferably in LaTeX. Students achieving at least half of the points on these homework assignments within the stipulated deadline will be allowed to register for an oral exam. The oral exam (typical duration: 30 minutes) accounts for 50% of the final grade and will take place at the end of the lecture period.

References

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