# Regularization based on Error Localization

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**Overview:**

**Notation:**

| | |
|---|---|
| $X$ | Complete normed real function space on $[\alpha, \beta]$ |
| $a \in X$ | Continuous function (multiplier) |
| $M_a \subset [\alpha, \beta]$ | Set of zeros of multiplier $a \in X$ (countable) |
| $\Lambda_a \in \mathcal{L}(X)$ | Linear bounded multiplication operator with $\Lambda_a x := ax$ |
| $K \in \mathcal{K}(X)$ | Linear compact operator on $X$ |

**Simplified Starting Problem ($\Lambda_a, X, X$):**

Let $X = (C[\alpha, \beta], \|.\|_\infty)$ or $X \in \{Y_p := L_p[\alpha, \beta] : 1 \leq p \leq \infty\}$.

$Tx = ax = y_\delta$ with $T = \Lambda_a \in \mathcal{L}(X)$ and $M_a \neq \emptyset$

Exact Data: $y_0 = Tx_0 \in T(X)$

Noisy Data: $y_\delta = y_0 + \Delta_\delta \in X$

Noise Level: $\delta \geq \|\Delta_\delta\|_X$

**Stabilized Approximations for $(\Lambda_a, X, X)$:**

Replace $Tx = ax = y_\delta$ with $T = \Lambda_a$ and $y_\delta = Tx_0 + \Delta_\delta$

by $S_\varepsilon(R_a)x_\varepsilon^\delta = R_a y_\delta$ with $S_\varepsilon(R_a) := \varepsilon I + R_a T$ and $\varepsilon > 0$.

**Case A:** Choose $R_a = r_a := a$ $\quad (S_\varepsilon(r_a) = \Lambda_{\varepsilon + a^2})$

Thus $x_0 - x_\varepsilon^\delta = \frac{\varepsilon x_0}{\varepsilon + a^2} - b_\varepsilon \Delta_\delta$ with $b_\varepsilon := \frac{a}{\varepsilon + a^2}$

**Case B:** Choose $R_a = \widehat{r}_a := sgn(a)$ $\quad (S_\varepsilon(\widehat{r}_a) = \Lambda_{\varepsilon + |a|})$

Thus $x_0 - x_\varepsilon^\delta = \frac{\varepsilon x_0}{\varepsilon + |a|} - \widehat{b}_\varepsilon \Delta_\delta$ with $\widehat{b}_\varepsilon := \frac{\widehat{r}_a}{\varepsilon + |a|}$

**Convergence w.r.t. Weighted Norm $\|a \cdot .\|_X$ for $(\Lambda_a, X, X)$:**

**Case A:** $R_a = r_a := a$ with $\varepsilon := \varepsilon(\delta; r_a) := \delta^2$ and $C := 1$:

$\|a(x_0 - x_\varepsilon^\delta)\|_X \leq C(\|x_0\|_X + 1)\delta$.

**Case B:** $R_a = \hat{r}_a := sgn(a)$ with $\varepsilon := \varepsilon(\delta; \hat{r}_a) := \delta$ and $C := 1$:

$\|a(x_0 - x_\varepsilon^\delta)\|_X \leq C(\|x_0\|_X + 1)\delta$.

**Error Localizing Property:**

$$\|a(x_0 - x_\varepsilon^\delta)\|_X \leq C(\|x_0\|_X + 1)\delta$$

for every $x_0, \Delta_\delta \in X$ with $\|\Delta_\delta\|_X \leq \delta$.

**Consequence:**

$$\|(x_0 - x_\varepsilon^\delta)\chi_{[\alpha,\beta]\setminus U}\|_X \leq \|a^{-1}\chi_{[\alpha,\beta]\setminus U}\|_\infty C(\|x_0\|_X + 1)\delta$$

for arbitrary neighb. $U \subset [\alpha, \beta]$ with $M_a \subset U$ and $card(M_a) < \infty$.

## Compactly Disturbed Multiplication Operators on $X$:

Now: $T := \Lambda_a - K \in \mathcal{L}(X)$ instead of $T = \Lambda_a \in \mathcal{L}(X)$.

**Aim:** Search for a subclass of $\mathcal{K}(X)$ (w.r.t. $a$) such that $S_\varepsilon(R_a)x_\varepsilon^\delta = R_a(Tx_0 + \Delta_\delta)$ with $S_\varepsilon(R_a) := \varepsilon I + R_a T$ yields

$$\|a(x_0 - x_\varepsilon^\delta)\|_X \leq C(\|x_0\|_X + 1)\delta$$

for every $x_0, \Delta_\delta \in X$ with $\|\Delta_\delta\|_X \leq \delta$.

I. Introduction

## II. Absolutely $a$-Compatible Operators

III. Regularization

**Absolutely $a$-Compatible Operators on $X$ (w.r.t. $R_a = r_a := a$):**

**Definition:** Let $T := \Lambda_a - K \in \mathcal{L}(X)$ with $K \in \mathcal{K}(X)$ and $X = (C[\alpha, \beta], \|.\|_\infty)$ or $X \in \{Y_p : 1 \leq p \leq \infty\}$. Then: $K \in \mathcal{K}_a^{abs}(X; r_a)$ iff:

$\exists$ const. $C \in \mathbb{R}^+$ and $\varepsilon_0 > 0$ such that

$$\|a(x_0 - x_\varepsilon^\delta)\|_X \leq C(\|x_0\|_X + 1)\delta$$

for all $x_0, \Delta_\delta \in X$ with $\|\Delta_\delta\|_X \leq \delta$ and $S_\varepsilon(r_a)x_\varepsilon^\delta = r_a(Tx_0 + \Delta_\delta)$, where $\varepsilon := \varepsilon(\delta; r_a) := \delta^2 \in (0, \varepsilon_0]$.

**Functional Analytic Characterization of $\mathcal{K}_a^{abs}(X;\, r_a)$:**

**Theorem:** Let $T := \Lambda_a - K \in \mathcal{L}(X)$ with $K \in \mathcal{K}(X)$ and $X = (C[\alpha,\beta], \|.\|_\infty)$ or $X \in \{Y_p : 1 \leq p \leq \infty\}$. Let $S_\varepsilon(r_a) := \varepsilon I + r_a T \in \mathcal{L}(X)$ be injective for $0 < \varepsilon \leq \varepsilon_0$.

Then, $K \in \mathcal{K}_a^{abs}(X; r_a)$ iff:

(1) $\kappa_1 := \sup\limits_{0<\varepsilon\leq\varepsilon_0} \|ab_\varepsilon[I - K\Lambda_{b_\varepsilon}]^{-1}\|_{X\to X} < \infty$ and

(2) $\kappa_2 := \sup\limits_{0<\varepsilon\leq\varepsilon_0} \|\sqrt{\varepsilon}b_\varepsilon[I - \Lambda_a K \Lambda^{-1}_{\varepsilon+a^2}]^{-1}\|_{X\to X} < \infty$,

where $b_\varepsilon := a/(\varepsilon + a^2)$.

**Remark:** One can find examples for $K \in \mathcal{K}_a^{abs}(X; r_a)$, where $K$ is finite or infinite dimensional, and where the maximal zero order $\nu(a)$ of the multiplier $a$ is smaller, equal to, or greater than $1$!

**Absolutely $a$-Compatible Operators on $X$ (w.r.t. $R_a = \hat{r}_a := sgn(a)$):**

**Definition:** Let $T := \Lambda_a - K \in \mathcal{L}(X)$ with $K \in \mathcal{K}(X)$ and $X \in \{Y_p : 1 \leq p \leq \infty\}$.

Then: $K \in \mathcal{K}_a^{abs}(X; \hat{r}_a)$ iff:

$\exists$ const. $C \in \mathbb{R}^+$ and $\varepsilon_0 > 0$ such that

$$\|a(x_0 - x_\varepsilon^\delta)\|_X \leq C(\|x_0\|_X + 1)\delta$$

for all $x_0, \Delta_\delta \in X$ with $\|\Delta_\delta\|_X \leq \delta$ and $S_\varepsilon(\hat{r}_a)x_\varepsilon^\delta = \hat{r}_a(Tx_0 + \Delta_\delta)$, where $\varepsilon := \varepsilon(\delta; \hat{r}_a) := \delta \in (0, \varepsilon_0]$.

**Functional Analytic Characterization of $\mathcal{K}_a^{abs}(X;\,\widehat{r}_a)$:**

**Theorem:** Let $T := \Lambda_a - K \in \mathcal{L}(X)$ with $K \in \mathcal{K}(X)$ and $X \in \{Y_p : 1 \le p \le \infty\}$.
Let $S_\varepsilon(\widehat{r}_a) := \varepsilon I + \widehat{r}_a T \in \mathcal{L}(X)$ be injective for $0 < \varepsilon \le \varepsilon_0$.

Then, $K \in \mathcal{K}_a^{abs}(X;\widehat{r}_a)$ iff:

$$\kappa := \sup_{0<\varepsilon\le\varepsilon_0} \|a\widehat{b}_\varepsilon[I - K\Lambda_{\widehat{b}_\varepsilon}]^{-1}\|_{X\to X} < \infty,$$

where $\widehat{b}_\varepsilon := \widehat{r}_a/(\varepsilon + |a|)$.

**Lavrentiev Resolvent Condition (LRC) for $a \geq 0$:**

Let $T := \Lambda_a - K \in \mathcal{L}(X)$ with $\sigma(T) \subset [0, \infty)$. Thus: $a \geq 0$ (nonnegative).

Then: $S_\varepsilon(\widehat{r}_a) := \varepsilon I + T \in \mathcal{L}(X)$ is injective for each $\varepsilon > 0$

and: $\varepsilon[\varepsilon I + T]^{-1} = \varepsilon \widehat{b}_\varepsilon [I - K\Lambda_{\widehat{b}_\varepsilon}]^{-1} \in \mathcal{L}(X)$.

$$\tilde{\kappa} := \sup_{0<\varepsilon\leq\varepsilon_0} \|\varepsilon[\varepsilon I + T]^{-1}\|_{X\to X} < \infty \quad \textbf{(LRC)}$$

**Connection between LRC and $K \in \mathcal{K}_a^{abs}(X;\hat{r}_a)$:**

Let $T := \Lambda_a - K \in \mathcal{L}(X)$ with $\sigma(T) \subset [0,\infty)$. Thus: $a \geq 0$.

$$K \in \mathcal{K}_a^{abs}(X;\hat{r}_a) \iff \sup_{0<\varepsilon\leq\varepsilon_0} \|a\hat{b}_\varepsilon[I - K\Lambda_{\hat{b}_\varepsilon}]^{-1}\|_{X\to X} < \infty$$

$$\sup_{0<\varepsilon\leq\varepsilon_0} \|\varepsilon[\varepsilon I + T]^{-1}\|_{X\to X} < \infty \iff \sup_{0<\varepsilon\leq\varepsilon_0} \|\varepsilon\hat{b}_\varepsilon[I - K\Lambda_{\hat{b}_\varepsilon}]^{-1}\|_{X\to X} < \infty.$$

Both Conditions together $\iff \sup_{0<\varepsilon\leq\varepsilon_0} \|[I - K\Lambda_{\hat{b}_\varepsilon}]^{-1}\|_{X\to X} < \infty.$

I. Introduction

II. Absolutely $a$-Compatible Operators

# III. Regularization

## Two-step Regularization by Error Localization and Smoothing

Let $T := \Lambda_a - K \in \mathcal{L}(X)$ with $a \in C[\alpha, \beta]$ for $X := (C[\alpha, \beta], \|.\|_\infty)$.
Moreover, let $card(M_a) < \infty$ with max. zero order $\nu := \nu(a) \in \mathbb{R}^+$.

Let: $K \in \mathcal{K}_a^{abs}(Y_\infty; R_a)$ with $R_a = r_a := a$ or $R_a = \hat{r}_a := sgn(a)$.

### 1. Step: Error Localization

Replace $Tx = y_\delta$ with $y_\delta = Tx_0 + \Delta_\delta \in X$

by $S_\varepsilon(R_a)x_\varepsilon^\delta = R_a y_\delta$

with $S_\varepsilon(R_a) := \varepsilon I + R_a T$ and $\varepsilon := \varepsilon(\delta; R_a) \in (0, \varepsilon_0]$.

Thus: $\|a(x_0 - x_\varepsilon^\delta)\|_X \leq C(\|x_0\|_X + 1)\delta$

**2. Step: Smoothing**

Smoothing the approximative solutions $x_\varepsilon^\delta$ in $\delta$-depending neighbourhoods of the zeros $t_i \in M_a$ by suitably defined Lagrangian interpolation polynomials (degree $\leq 2n-1$) yields smoothed approximative solutions $z^\delta_{\gamma(\delta;R_a)}$.

Regularization parameter vector: $\gamma(\delta; R_a) = (\gamma_i(\delta; R_a))_i$ (depending on $\delta, \nu_i$ and $n$)

If $x_0 \in C^{2n}[\alpha, \beta]$, one gets for $R_a = a$ or $R_a = sgn(a)$:

$$\|x_0 - z^\delta_{\gamma(\delta;R_a)}\|_X = O(\delta^{\frac{2n}{2n+\nu}}) \quad \text{for } \delta \downarrow 0.$$

# Appendix

**Example:**

Diff. Equation: $t^4 x''(t) + 8t^3 x'(t) + (t^3 + 12t^2 - 2t)x(t) = \hat{y}_0(t)$

with: $\hat{y}_0(t) := 6t^2 \cos(t) - (t^3 - t^2 - 6t + 2)\sin(t), \quad t \in [-2\pi, 2\pi]$

Solution: $x_0(t) = \sin(t)/t$ with the BV: $x(-2\pi) = 0 = x(2\pi)$

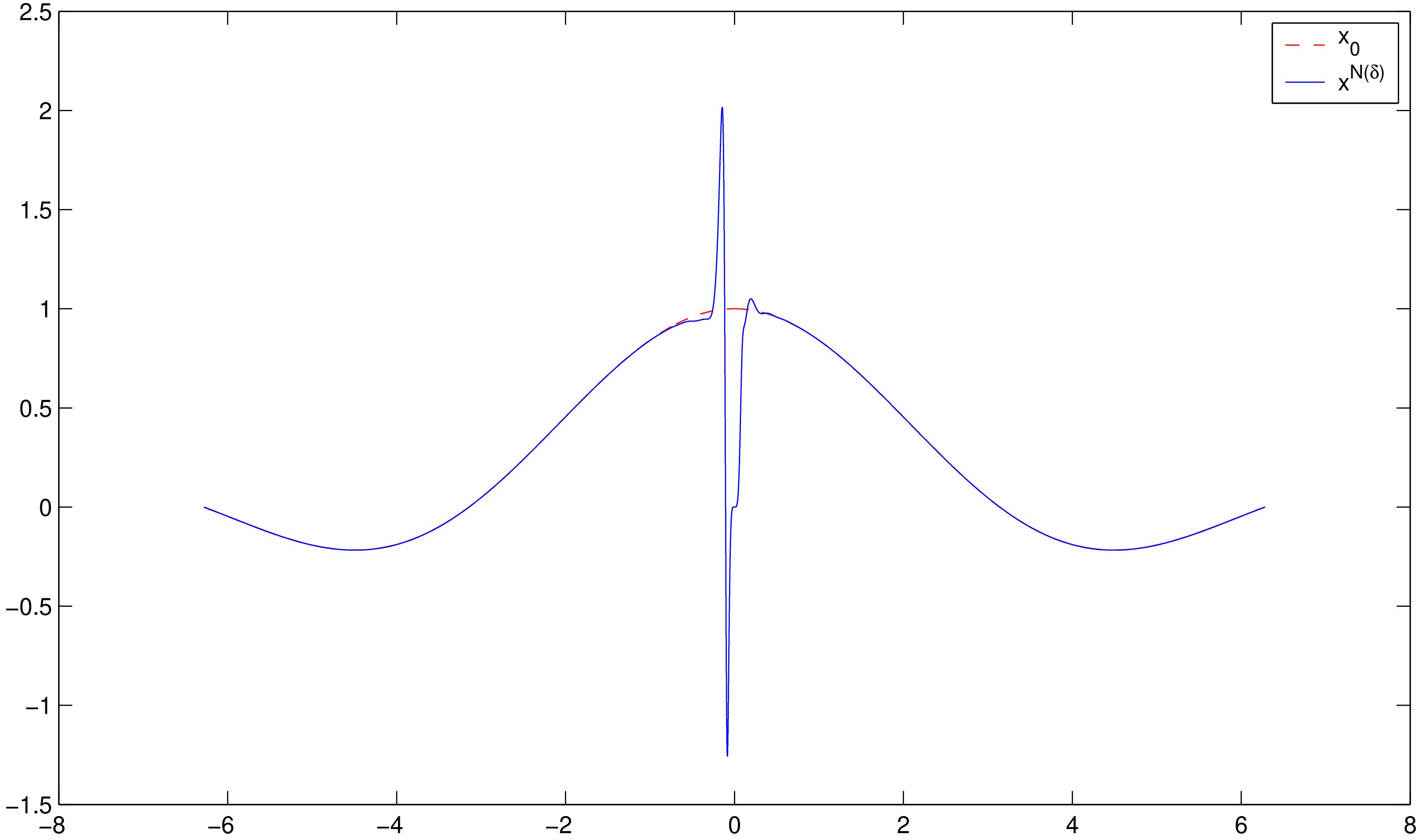
Equiv. Third-Kind IE: $t^4 x(t) - \int_{-2\pi}^{2\pi} (\tau^3 - 2\tau) g_{-2\pi,2\pi}(t,\tau) x(\tau) d\tau = y_0(t)$

Undisturbed Data: $y_0(t) = (t^3 - t^2 + 8)\sin(t) - 4t\cos(t) + 4t$

Disturbed Data: $y_\delta(t) := y_0(t) + \Delta_\delta(t)$

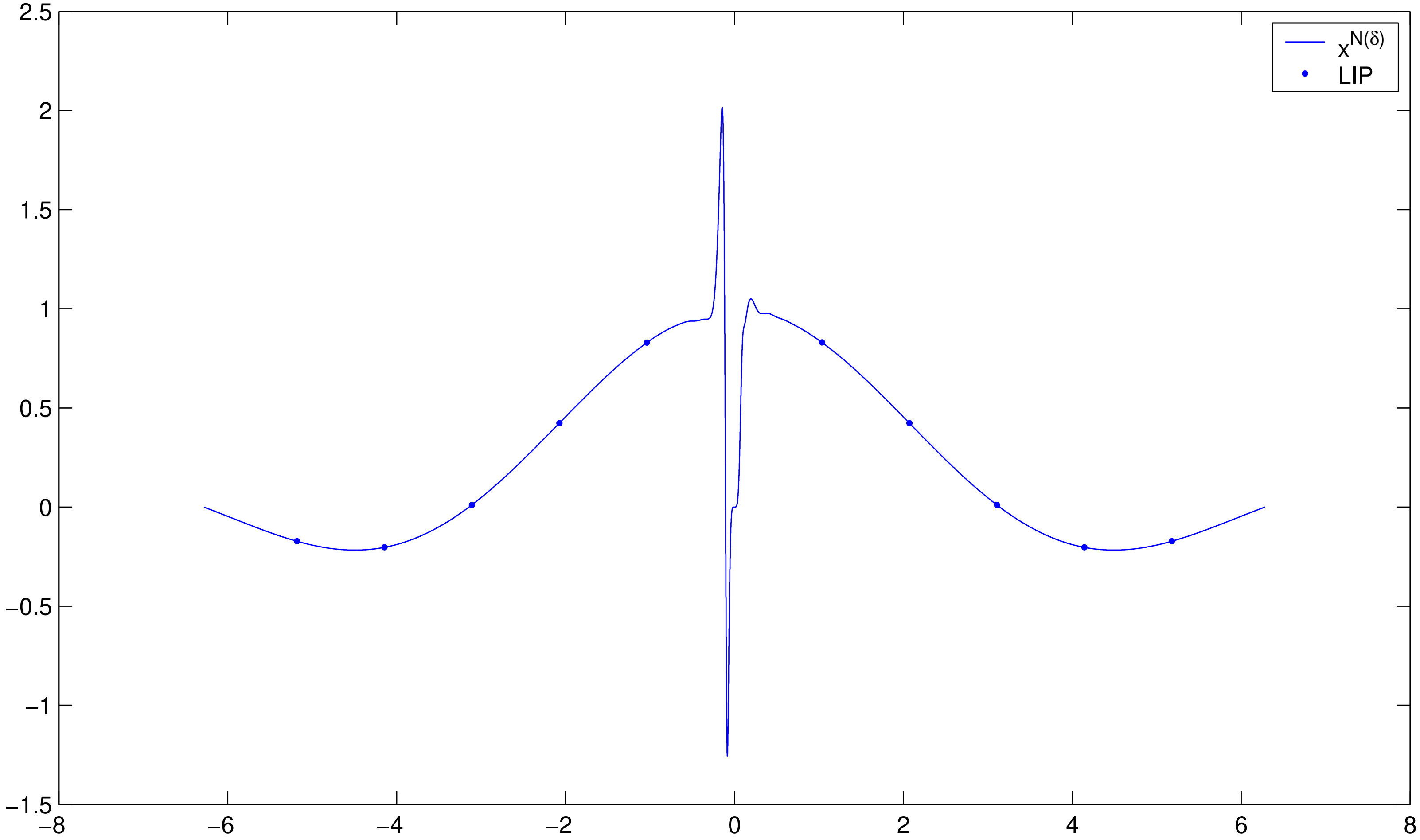
with: $\Delta_\delta(t) := \delta \cdot \sin(\frac{\pi}{4} + 10\pi t)$.

Example with $\delta = $ 1e-4



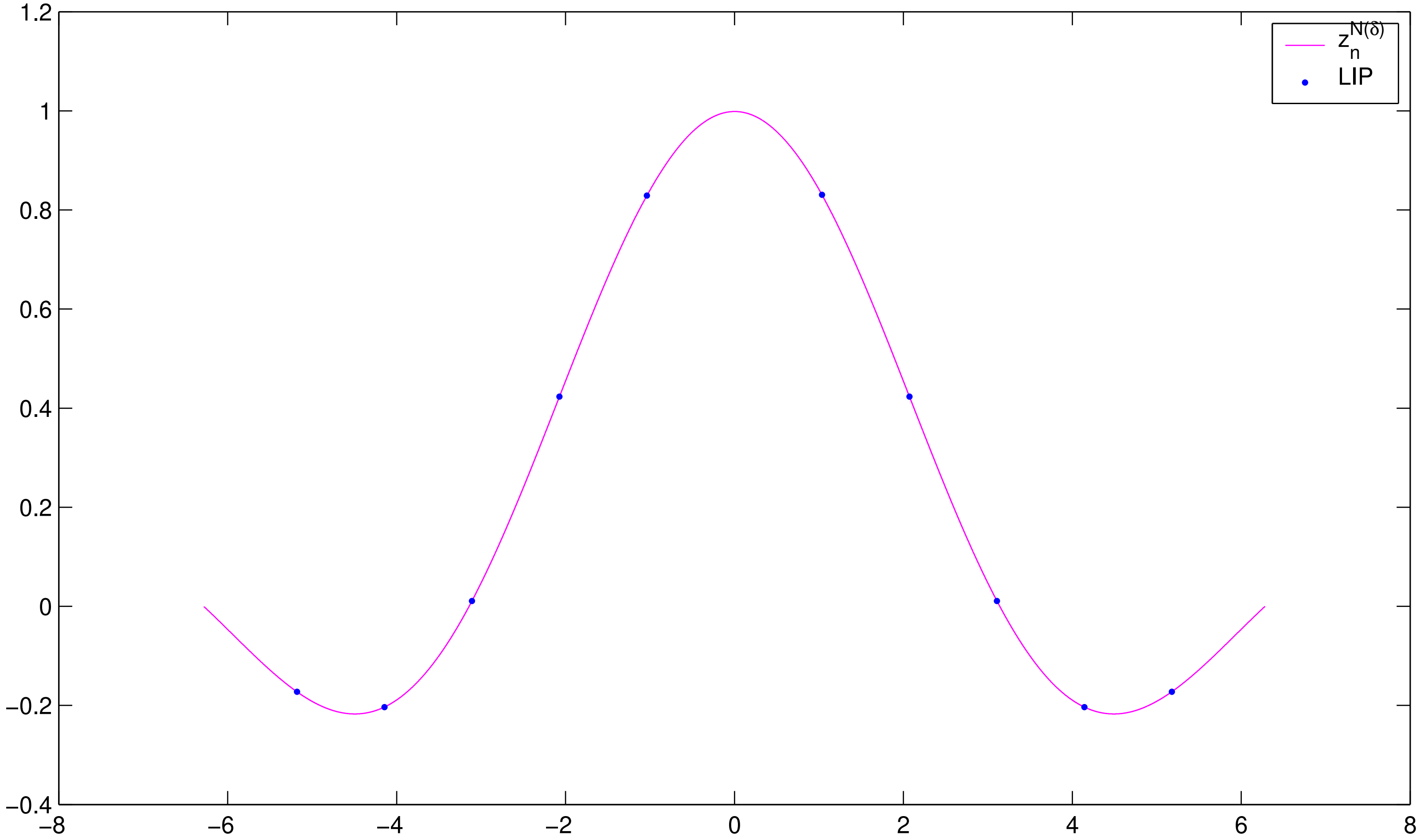

Approximative Solution (AS) $x^{N(\delta)}$

## Example with $\delta = 1\text{e-}4$ and $n = 5$



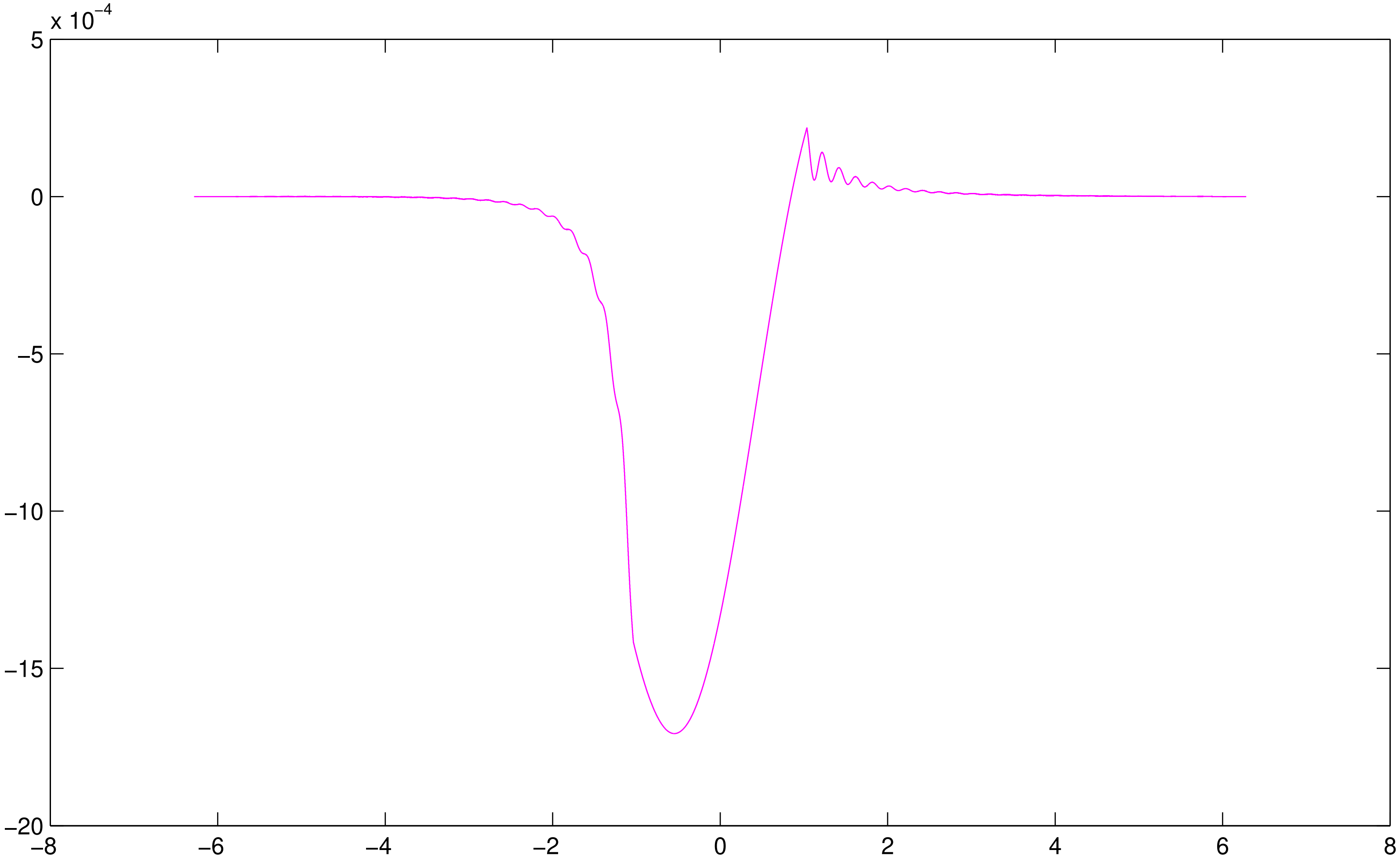

Approximative Solution (AS) $x^{N(\delta)}$ with LIP

## Example with $\delta = 1\text{e-}4$ and $n = 5$




Smoothed Approximative Solution (SAS) $z_n^{N(\delta)}$

Example with $\delta = 1\text{e-}4$ and $n = 5$




Error Function (EF) $z_n^{N(\delta)} - x_0$ Scale: $\times 10^{-4}$