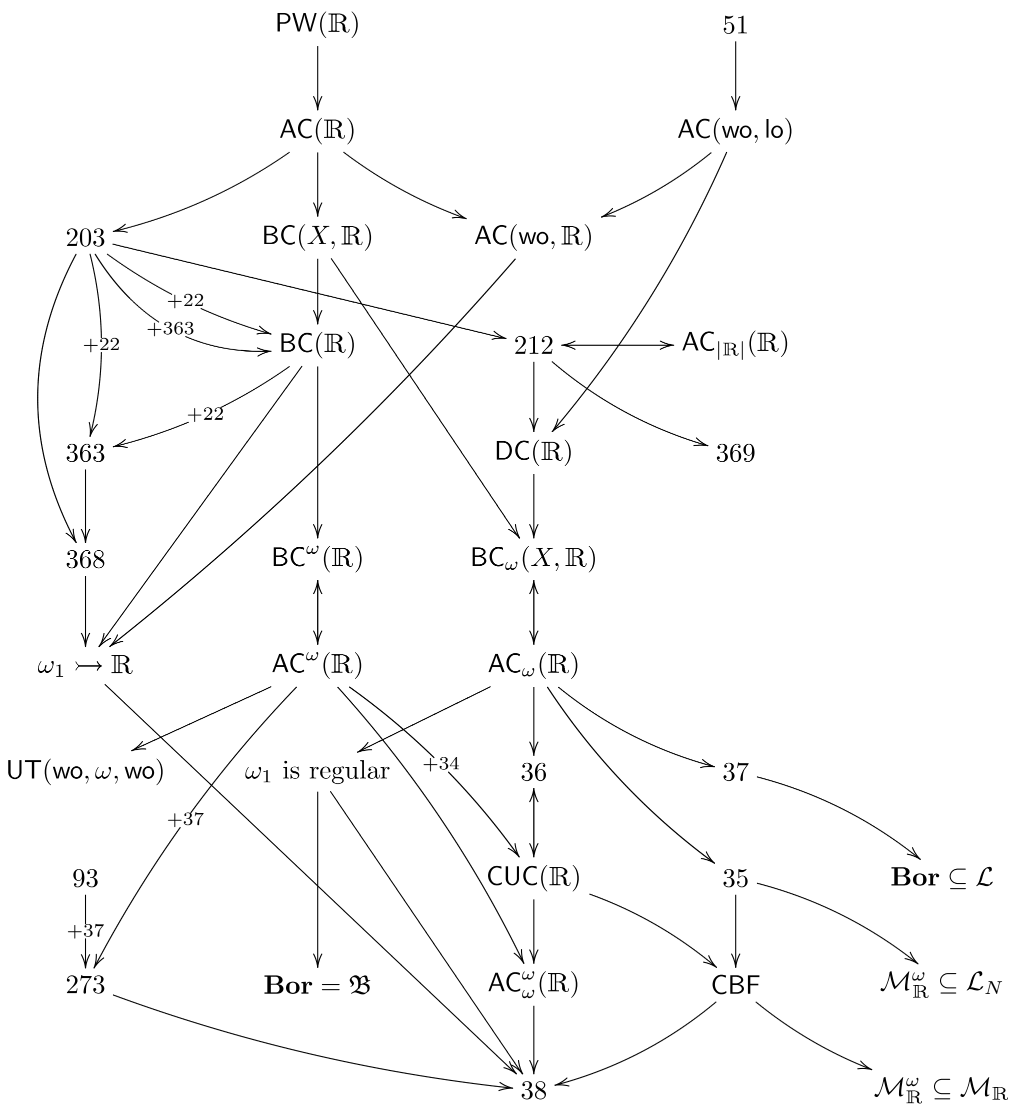
The diagram


PW(ℝ)
51
AC(ℝ)
AC(wo, lo)
203
BC(X, ℝ)
AC(wo, ℝ)
+22
+363
BC(ℝ)
212
AC_{|ℝ|}(ℝ)
+22
+22
363
DC(ℝ)
369
368
BC^ω(ℝ)
BC_ω(X, ℝ)
ω_1 ↣ ℝ
AC^ω(ℝ)
AC_ω(ℝ)
UT(wo, ω, wo)
ω_1 is regular
+34
36
37
+37
93
CUC(ℝ)
35
Bor ⊆ ℒ
+37
273
Bor = 𝔅
AC^ω_ω(ℝ)
CBF
ℳ^ω_ℝ ⊆ ℒ_N
38
ℳ^ω_ℝ ⊆ ℳ_ℝ

1 What do the form names mean?

- **Bor** is the Borel σ-algebra of $\mathbb{R}$, i.e., the intersection of all σ-algebras containing the open subsets of $\mathbb{R}$.
- $\mathcal{N}_{\mathbb{R}}$ is the ideal of all nowhere dense subsets of $\mathbb{R}$,
- $\mathcal{M}_{\mathbb{R}} = \{\bigcup Q \in \wp(\mathbb{R})\ ;\ Q \in [\mathcal{N}_{\mathbb{R}}]^{\leq\omega}\}$ which is the ideal of all meager subsets of $\mathbb{R}$,
- $\mathcal{N}^{c}_{\mathbb{R}}$ is the ideal of all closed nowhere dense subsets of $\mathbb{R}$,
- $\mathcal{M}^{c}_{\mathbb{R}} = \{\bigcup Q \in \wp(\mathbb{R})\ ;\ Q \in [\mathcal{N}^{c}_{\mathbb{R}}]^{\leq\omega}\}$,
- $\mathcal{M}^{\omega}_{\mathbb{R}} = \{\bigcup Q \in \wp(\mathbb{R})\ ;\ Q \in [[\mathbb{R}]^{\leq\omega}]^{\leq\omega}\}$ which is the ideal of all countable unions of countable subsets of $\mathbb{R}$,
- $\mathcal{BF} = \{\mathcal{F} \subset \mathbb{N}^{\mathbb{N}}\ ;\ \mathcal{F}$ is dominated.$\} = \{\mathcal{F} \subset \mathbb{N}^{\mathbb{N}}\ ;\ \exists g \in \mathbb{N}^{\mathbb{N}} \forall f \in \mathcal{F}, f(n) \leq g(n)$ for almost all $n \in \mathbb{N}\}$,
- μ^* is the outer measure that when it's assumed to be countably subadditive it's possible to define the Lebesque measure μ on the measurable sets.
- $\mathcal{L}$ is the class of all measurable subsets of $\mathbb{R}$ and
- $\mathcal{L}_{\mathcal{N}} = \{x \in \wp(\mathbb{R})\ ;\ \mu^*(x) = 0\}$ which is the ideal of outer Lebesgue measure zero subsets of $\mathbb{R}$.

We also use special notation for statements as the Axiom of Choice and weaker forms of it, as follows.

- $\mathsf{AC}(\mathbb{R})$ is the Axiom of Choice restricted to the real numbers,
- $\mathsf{DC}(\mathbb{R})$ is the Axiom of Dependent Choices,
- $\mathsf{AC}_{\omega}(\mathbb{R})$ is AC restricted to countable families of subsets of the reals.[1],
- $\mathsf{AC}^{\omega}(\mathbb{R})$ is AC restricted to families of countable subsets of $\mathbb{R}$,
- $\mathsf{AC}^{\omega}_{\omega}(\mathbb{R})$ is AC restricted to countable families of countable subsets of $\mathbb{R}$,
- $\mathsf{CUC}(\mathbb{R})$ is the statement "A countable union of countable subsets of $\mathbb{R}$ is countable",
- CBF is the statement "For every family $\mathcal{F} = \{F_i \subset \mathbb{N}^{\mathbb{N}}\ ;\ i \in \omega\}$ of non-empty countable sets, there is a set $H = \{h_i \in \mathbb{N}^{\mathbb{N}}\ ;\ i \in \omega\}$ such that $\forall i \in \omega \forall f \in F_i, f(n) \leq h_i(n)$ for almost all $n \in \omega$",
- $\mathsf{BC}(\mathbb{R})$ is the statement "**Bor** $\setminus \{\varnothing\}$ has a choice function",
- $\mathsf{BC}_{\omega}(\mathbb{R})$ is BC restricted to countable families of subsets of **Bor**,
- $\mathsf{BC}^{\omega}(\mathbb{R})$ is BC restricted to families of countable subsets of **Bor**,
- $\mathsf{UT}(\mathsf{wo}, \omega, \mathsf{wo})$ is the statement "Every well ordered union of countable subsets of $\mathbb{R}$ can be well ordered",

[1] in general we will use subscript ω for restriction to countable families and superscript ω for restriction to countable subsets

- $\mathsf{BC}(X,\mathbb{R})$ is the statement "For $X \subset \mathbb{R}$, the Borel algebra $\mathcal{B}_X$ of X with the relative topology has a choice function",
- $\mathsf{BC}_\omega(X,\mathbb{R})$ is $BC(X,\mathbb{R})$ restricted to countable families,
- $\mathsf{AC}(\mathsf{wo},\mathbb{R})$ is the statement "For every family $\mathcal{A} = \{A_i \,;\, i \in \mu\}$, with $\mu \in \mathrm{Ord}$, of non-empty subsets of $\mathbb{R}$, there is a $c = \{c_i \,;\, i \in \mu\}$ choice set" and
- $\mathsf{AC}(\mathsf{wo},\mathsf{lo})$ is the statement "For every well ordered family $\mathcal{A}$ of k-many sets, $k \in \mathrm{Ord}$, such that $\bigcup \mathcal{A}$ is linearly ordered, $\mathcal{A}$ has a choice set",

Moreover, the following form numbers represent the following forms.

13 - Every infinite subset of $\mathbb{R}$ has a countably infinite subset.

22 - If $A \subset \mathbb{R}$ such that $|A| = 2^{\aleph_0}$ and $\forall x \in A, |x| = 2^{\aleph_0}$, then $|\bigcup A| = 2^{\aleph_0}$.

34 - $\aleph_1$ is regular.

35 - The union of countably many meager sets is meager.

36 - If $A \subseteq \mathbb{R}^n$ and $A \cap B$ is countable for every bounded B, then A is countable.

37 - The Lebesque measure is countably additive.

38 - $\mathbb{R}$ is not a countable union of countable sets.

51 - All linear orderings have a cofinal sub well ordering.

93 - $\mathcal{L} \subsetneq \wp(\mathbb{R})$.

170 - $\aleph_1 \leq 2^{\aleph_0}$, where by for two sets A, B, $A \leq B$ is defined as 'there is an injection from A to B.

203 - Every partition of $\mathbb{R}$ into non-empty subsets has a choice function

212 - $AC_{|\mathbb{R}|}(\mathbb{R})$, i.e., AC restricted to families of size $|\mathbb{R}|$ of arbitrary subsets of $\mathbb{R}$.

273 - $\mathbf{Bor} \neq \wp(\mathbb{R})$.

363 - $|\mathbf{Bor}| = 2^{\aleph_0}$.

368 - $|\{x \subseteq \mathbb{R} \,;\, |x| \leq \aleph_0\}| = 2^{\aleph_0}$.

369 - If $\mathbb{R}$ is partitioned into two sets, one of them has cardinality $2^{\aleph_0}$.

Finally, let's define the classes $\boldsymbol{\Sigma}^0_\alpha, \boldsymbol{\Pi}^0_\alpha, \boldsymbol{\Delta}^0_\alpha, \mathfrak{B} \subseteq \wp(\mathbb{R})$ inductively as

$$\begin{aligned}
\boldsymbol{\Sigma}^0_1 &:= \{V \subseteq \mathbb{R} \,;\, V \text{ is open}\}, \\
\text{for } \alpha > 1,\ \boldsymbol{\Sigma}^0_\alpha &:= \{\bigcup_n X_n \,;\, \exists\langle \beta_n \,;\, n \in \omega\rangle (1 \leq \beta_n < \alpha \text{ and } X_n \in \boldsymbol{\Pi}^0_\alpha)\}, \\
\text{for } \alpha > 1,\ \boldsymbol{\Pi}^0_\alpha &:= \{\mathbb{R} \setminus X \,;\, X \in \boldsymbol{\Sigma}^0_\alpha\} \\
\text{for } \alpha \geq 1,\ \boldsymbol{\Delta}^0_\alpha &:= \boldsymbol{\Sigma}^0_\alpha \cap \boldsymbol{\Pi}^0_\alpha \\
\text{and } \mathfrak{B} &:= \bigcup_{\alpha < \omega_1} \boldsymbol{\Sigma}^0_\alpha
\end{aligned}$$