TUTORIAL ON SYMMETRIC MODELS AND THE APPROXIMATION LEMMA

IOANNA M. DIMITRÍOU

These notes describe the construction of a symmetric model when forcing with partial orders. The construction is described also in [Jec03, pages 249–261 of Ch.15 and pages 221–223 of Ch.14], in the first edition of this book [Jec78] in more detail, and fully in [Jec73]. In all these sources this construction is discussed for forcing with Boolean valued models. Nowadays most people are more familiar with forcing in terms of partial orders and this note covers this audience. Moreover, it has a comprehensive guide to proving the approximation lemma for a symmetric model. Most probably the symmetric models you'll ever encounter will satisfy this useful lemma. It basically says that the sets of ordinals in the symmetric model can be approximated by generic extensions of some "initial part" of the forcing.

By understanding the text here and completing the exercises you'll have all the tools necessary to work with any symmetric model. The prerequisites to understand this note is knowledge of basic set theory and basic forcing. Kunen's [Kun80] up to Chapter VII is sufficient. Knowledge of large cardinals isn't necessary as we'll define them when we use them.

The author thanks her student Rogier Jacobs for following this course and giving valuable input to the previous version of these notes.

1. A little history

The method of taking symmetric submodels of a generic extension is used to create models where the axiom of choice (AC) does not hold everywhere. This method is inspired by the older method of permutation models, by Abraham Fraenkel and others, over 75 years ago. In that method, the base model is a model of ZFCA, i.e., ZFC with the axiom for the existence of atoms added. Atoms have the same defining property of the empty set but are not equal to it. Therefore atoms are excluded from the axiom of extensionality.

Date: March 11, 2010.

These atoms can be as many as we want (by adding the axiom "the set of atoms is κ big") and they are indistinguishable from each other.

By permuting these atoms we can construct models of $\mathsf{ZFA} + \neg\mathsf{AC}$. The similarity between atoms in ZFCA and parts of the generic object in a forcing construction was probably noted by set theorists after the introduction of forcing. So the technique of permutation models was adapted to fit the forcing constructions and became the symmetric model technique. The use of symmetry arguments goes back to Abraham Fraenkel ([Fra22]). A little later, Andrzej Mostowski ([Mos39]) and Ernst Specker ([Spe57]) developed a theory of this technique. Paul Cohen used these arguments with his forcing technique ([Coh63], [Coh64]), and Thomas Jech ([Jec71]) and Dana Scott (unpublished) formulated the technique in terms of Boolean valued models, thus making it accessible to every student of set theory.

Roughly, we use an automorphism group of the partial order and a normal filter over that group. We extend the automorphisms on the conditions to automorphisms on the names and in the new model we allow only these names that (hereditarily) remain intact from the permutations from a set in the filter. These names are called (hereditarily) symmetric and the class of their interpretations a symmetric model.

2. Short forcing reminder and notation

Before we go into the details of symmetric models, we give a short reminder of the forcing technique and some words on the notation used here. A detailed and popular introduction to forcing is Kunen's book, [Kun80]. Metamathematical concerns about this method will not be addressed here, see [Kun80, Ch.VII].

Our background theory here is ZFC. Fix V a model of ZFC and $\mathbb{P} \in V$ a partial order[1] with maximal element $\mathbb{1}$. For two elements p, q of the partial order we say that p is stronger than q when $p \leq q$. To do a forcing construction we need to have a $\mathbb{P}$-generic filter on the partial order $\mathbb{P}$. Such a G is a filter on the partial order that intersects all the dense subsets of $\mathbb{P}$ in V. Usually G is not in the ground model V. Intuitively to get a forcing extension we close $V \cup \{G\}$ under all set theoretic operations. Formally

[1]By that we mean that $\mathbb{P}$ is a set with a relation $\leq \subseteq \mathbb{P} \times \mathbb{P}$ that is reflexive, antisymmetric, and transitive.

though, we take all objects that are definable in a way from G and finitely many elements of V. We do that by using names for the new objects, names that are definable in V.

Names are defined by recursion on the rank, as sets of pairs in V such that the first coordinate is an already defined name and the second coordinate is an element of the partial order. We write $V^{\mathbb{P}}$ for the class of $\mathbb{P}$-names in a ground model V.

We use both standard notations for names; $\dot{x}$ for a name of an already given x and σ, τ, etc. for arbitrary names. For $x \in V$ we denote its canonical name by $\check{x}$. Canonical names for elements of V are defined recursively as

$$\check{x} \stackrel{\text{def}}{=} \{(\check{y}, \mathbb{1}) \ ; \ y \in x\}.$$

Canonical names for sets of elements of V are names that consist of pairs $(\check{\alpha}, p)$ for $p \in \mathbb{P}$. When G is a $\mathbb{P}$-generic filter over V and τ is a name, we write τ^G for the interpretation (valuation) of τ according to G, which is defined as

$$\tau^G \stackrel{\text{def}}{=} \{\sigma^G \ ; \ \exists p \in G, \ (\sigma, p) \in \tau\}.$$

The generic forcing extension is defined as $V[G] \stackrel{\text{def}}{=} \{\tau^G \ ; \ \tau \in V^{\mathbb{P}}\}$, it is a transitive model of ZFC, and it is the smallest ZFC model that contains both V and G.

Definition 2.1. The forcing language $\mathcal{L}_F$ is a first order language that contains the $\in$-relation and all names as constants. For a formula $\varphi \in \mathcal{L}_F$ one could write φ^G for the φ with all its constants (names) valuated in $V[G]$ and its bounded quantifiers ranging over $V[G]$.

Definition 2.2. Define a relation $\Vdash$ between the conditions in $\mathbb{P}$ and the sentences of $\mathcal{L}_F$ as

$$\begin{aligned} p \Vdash \varphi(\vec{\tau}) \Longleftrightarrow & \text{For every } V\text{-generic filter } G \text{ on } \mathbb{P} \text{ such that } p \in G, \\ & V[G] \models \varphi^G(\vec{\tau}^G). \end{aligned}$$

Note that this is a definition in our outer model, not in V. As one will read in [Kun80, Ch.VII, Definition 3.3 and Theorem 3.6], this $\Vdash$ is definable in V as well, and the two definitions are equivalent. The following is referred to as "the forcing theorem".

Theorem 2.3. Let $\langle \mathbb{P}, \leq \rangle$ be a partial order in the ground model V. If φ is a formula of $\mathcal{L}_F$ with n free variables then for every G that is a $\mathbb{P}$-generic filter and all $\mathbb{P}$-names $\tau_1, \ldots, \tau_n$,

$$V[G] \models \varphi^G(\tau_1^G, \ldots, \tau_n^G) \Longleftrightarrow \exists p \in G(p \Vdash \varphi(\tau_1, \ldots, \tau_n)).$$

Finally we state the following useful properties of the forcing relation.

Proposition 2.4. Let $\mathbb{P}$ be a partial order in V and φ, ψ arbitrary sentences in the forcing language for $\mathbb{P}$. Some of the most important properties of the forcing relation are the following:

(a) $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$.
(b) There is no p such that $p \Vdash \varphi$ and $p \Vdash \neg\varphi$).
(c) For every p there is a $q \leq p$ such that $q \Vdash \varphi$ or $q \Vdash \neg\varphi$ (we say q decides φ).
(d) $p \Vdash \neg\varphi \Longleftrightarrow$ there is no $q \leq p$ such that $q \Vdash \varphi$.
(e) $p \Vdash \varphi \wedge \psi \Longleftrightarrow p \Vdash \varphi$ and $p \Vdash \psi$.
$p \Vdash \forall x \varphi \Longleftrightarrow$ for every $\tau \in V^{\mathbb{P}}$, $p \Vdash \varphi(\tau)$.
(f) $p \Vdash \varphi \vee \psi \Longleftrightarrow \forall q \leq p \exists r \leq q(r \Vdash \varphi$ or $r \Vdash \psi)$.
$p \Vdash \exists x \varphi \Longleftrightarrow \forall q \leq p \exists r \leq q \exists \dot{a} \in V^{\mathbb{P}}(r \Vdash \varphi(\dot{a}))$.
(g) $p \Vdash \exists x \varphi \Rightarrow$ for some $\tau \in V^{\mathbb{P}}$, $p \Vdash \varphi(\tau)$.

We are not going to make a distinction between φ and φ^G, since it will be always clear from the context which one we mean.

Lastly, two important facts about forcing. Let ρ be a cardinal and $\mathbb{P}$ a partial order. If $\mathbb{P}$ has the ρ-chain condition (ρ-cc), i.e., of all its antichains have cardinality $< \rho$, then $\mathbb{P}$ preserves all cardinals and cofinalities $\geq \rho$ ([Kun80, Ch.VII, Lemma 6.9]). If $\mathbb{P}$ is ρ-closed, i.e., if for every ρ-long increasing sequence of conditions in $\mathbb{P}$ $\langle p_\alpha \; ; \; \alpha < \rho \rangle$ there is a condition $p \in \mathbb{P}$ stronger than all the p_α's, then $\mathbb{P}$ preserves cardinals and cofinalities $\leq \rho$. Now we have all we need to build symmetric models.

3. How to construct a symmetric model

We give a simple and quick presentation of the technique of creating symmetric models in terms of forcing with partial orders. This is initially just a translation to partial orders of the standard technique that Jech presents in his [Jec03] (for forcing with Boolean values). Moreover we introduce several notions that lead to the definition of a triple $(\mathbb{P}, \mathcal{G}, I)$ with the approximation property. Symmetric forcing with such a triple ensures that in the resulting symmetric model the sets of ordinals are very well behaved.

As we define symmetric models we'll have the following example in our minds. Let

$$\mathbb{F} \stackrel{\text{def}}{=} \{p : \omega \times \omega \rightharpoonup \aleph_\omega \, ; \, |p| < \omega \text{ and } \forall (n,i) \in \mathsf{dom} p,\ p(n,i) < \omega_n\}$$

be ordered by reverse inclusion, i.e., $p \leq q$ iff $p \supseteq q$. This partial order is used to build the well known Feferman-Lévy model, first constructed in 1963 (for the abstract see [FL63]). In this model, the reals are a countable union of countable sets and therefore both AC and AD fail.

Exercise 1. *Prove that for every $n \in \omega$, the partial order $\mathbb{F}$ adds a countable set whose elements are surjective functions from ω to ω_n (we call this a set of collapsing functions for ω_n).*

In any generic extension the ordinal $\kappa = \aleph_\omega^V$ has become a countable union of countable sets and therefore is countable.

If $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ is a partial order, an automorphism a of $\mathbb{P}$ is a bijection of $\mathbb{P}$ to itself which preserves $\leq$ and $\mathbb{1}$ both ways. If a is an automorphism of $\mathbb{P}$, then define by recursion on $V^{\mathbb{P}}$,

$$a_*(\tau) \stackrel{\text{def}}{=} \{(a_*(\sigma), a(p)) \, ; \, (\sigma, p) \in \tau\}.$$

Given a, we will denote a_* also by a as it will be clear from the context what we mean. We'll need to use an automorphism group $\mathcal{G}$ of our partial order.

For our running example consider $\mathcal{G}_{\mathbb{F}}$ to be the full permutation group of ω. Extend $\mathcal{G}_{\mathbb{F}}$ to an automorphism group of $\mathbb{F}$ by letting an $a \in \mathcal{G}_{\mathbb{F}}$ act on a $p \in \mathbb{F}$ by

$$a^*(p) \stackrel{\text{def}}{=} \{(n, a(i), \beta) \, ; \, (n, i, \beta) \in p\}.$$

We'll identify a^* with $a \in \mathcal{G}_{\mathbb{F}}$. It's easy to check that this is indeed an automorphism group of $\mathbb{F}$.

The following is called the "symmetry lemma".

Lemma 3.1. Let $\mathbb{P}$ be a partial order and $\mathcal{G}$ an automorphism group of $\mathbb{P}$. Let φ be a formula of the forcing language with n free variables and let $\tau_1, \ldots, \tau_n \in V^{\mathbb{P}}$ be names. If $a \in \mathcal{G}$ then

$$p \Vdash \varphi(\tau_1, \ldots, \tau_n) \iff a(p) \Vdash \varphi(a(\tau_1), \ldots, a(\tau_n)).$$

Exercise 2. *Prove the symmetry lemma. (Hint: Induction and the properties of the forcing relation.)*

Definition 3.2. Let $\mathcal{G}$ be a group and $\mathcal{F} \subseteq \mathcal{P}(\mathcal{G})$ a filter over $\mathcal{G}$. We say that $\mathcal{F}$ is a normal filter if for every $K \in \mathcal{F}$ and every $a \in \mathcal{G}$, the conjugate aKa^{-1} is in $\mathcal{F}$.

In some sources, a "filter" over a group should always be a normal filter (e.g., in [Jec03, page 251, (15.34)]).

For every $n \in \omega$ define the following sets.

$$E_n \stackrel{\text{def}}{=} \{p \cap (\omega \times n \times \omega_n) \;;\, p \in \mathbb{F}\}$$
$$\mathsf{fix} E_n \stackrel{\text{def}}{=} \{a \in \mathcal{G}_{\mathbb{F}} \;;\, \forall p \in E_n (ap = p)\}, \text{ and}$$
$$\mathcal{F}_{\mathbb{F}} \stackrel{\text{def}}{=} \{X \subseteq \mathcal{G}_{\mathbb{F}} \;;\, \exists n \in \omega,\ \mathsf{fix} E_n \subseteq X\}.$$

The set $\mathcal{F}_{\mathbb{F}}$ is a normal filter over $\mathcal{G}$ (See Exercise 4).

For the rest of this section fix a partial order $\mathbb{P}$, an automorphism group $\mathcal{G}$ of $\mathbb{P}$, and a normal filter $\mathcal{F}$ over $\mathcal{G}$.

Definition 3.3. For each $\tau \in V^{\mathbb{P}}$, we define its symmetry group with respect to $\mathcal{G}$ as

$$\mathsf{sym}^{\mathcal{G}} \tau \stackrel{\text{def}}{=} \{a \in \mathcal{G} \;;\, a\tau = \tau\}.$$

If we see $\mathcal{G}$ as an automorphism group of $V^{\mathbb{P}}$ then for a name τ, $\mathsf{sym}^{\mathcal{G}} \tau$ is the stabilizer group of τ. We say that τ is symmetric if $\mathsf{sym}^{\mathcal{G}} \tau \in \mathcal{F}$. We denote by $\mathsf{HS}^{\mathcal{F}}$ the class of all hereditarily symmetric names, i.e.,

$$\mathsf{HS}^{\mathcal{F}} \stackrel{\text{def}}{=} \{\tau \in V^{G} \;;\, \forall \sigma \in \mathsf{tc}_{\mathsf{dom}}(\tau),\ \mathsf{sym}^{\mathcal{G}} \sigma \in \mathcal{F}\},$$

where $\mathsf{tc}_{\mathsf{dom}}(\tau)$ is defined as the union of all x_n, which are defined recursively by $x_0 \stackrel{\text{def}}{=} \mathsf{dom}\tau$ and $x_{n+1} \stackrel{\text{def}}{=} \bigcup \{\mathsf{dom}\sigma \;;\, \sigma \in x_n\}$.

When it's clear from the context we'll denote these newly defined notions by sym, HS. All canonical names are hereditarily symmetric:

Lemma 3.4. If a is an automorphism of $\mathbb{P}$ then for every canonical name $\check{x} \in V^{\mathbb{P}}$, $a(\check{x}) = \check{x}$.

The proof of this lemma is a simple induction using the definition of $\check{x}$.

Definition 3.5. We define the symmetric submodel of $V[G]$ with respect to $\mathcal{F}$ by

$$V(G)^{\mathcal{F}} \overset{\text{def}}{=} \{\tau^G \,;\, \tau \in \mathsf{HS}^{\mathcal{F}}\}.$$

We'll often denote a symmetric model by simply $V(G)$. To talk about the truth of formulas in a symmetric model we use the symmetric forcing relation.

Definition 3.6. For a formula φ and names $\vec{\tau}$ in HS, define the relation $\Vdash_{\mathsf{HS}}$ by

$$p \Vdash_{\mathsf{HS}} \varphi(\vec{\tau}) \overset{\text{def}}{\iff} \text{ for any } \mathbb{P}\text{-generic filter } G, \text{ and any } p \in G,\ V(G) \models \varphi$$

This "symmetric forcing" relation has the same properties as the usual forcing relation $\Vdash$ and the symmetry lemma holds for it too.

Theorem 3.7. A symmetric model $V(G)^{\mathcal{F}}$ is a transitive model of ZF and $V \subseteq V(G)^{\mathcal{F}} \subseteq V[G]$.

Proof. That $V \subseteq V(G)^{\mathcal{F}} \subseteq V[G]$ is obvious and by the heredity of HS, we get that $V(G)^{\mathcal{F}}$ is transitive. Extensionality, foundation, empty set, and infinity hold because $V \subseteq V(G)^{\mathcal{F}}$ and $V(G)^{\mathcal{F}}$ is transitive. For the separation schema let φ be a formula and let $y = \{x \in z \,;\, V(G)^{\mathcal{F}} \models \varphi(x, w)\}$ where $z, w \in V(G)^{\mathcal{F}}$ with $\dot{z}, \dot{w}$ their HS names respectively. Define a name for y as follows.

$$\dot{y} \overset{\text{def}}{=} \{(\sigma, p) \,;\, \sigma \in \mathsf{dom}(\dot{z}) \text{ and } p \Vdash_{\mathsf{HS}} \varphi(\sigma, \dot{w})\},$$

This is a HS name for y because for every $a \in \mathsf{sym}\dot{z} \cap \mathsf{sym}\dot{w}$, we have that

$$\begin{aligned} a(\dot{y}) &= \{(a(\sigma), a(p)) \,;\, a\sigma \in \mathsf{dom}\dot{z} \text{ and } a(p) \Vdash_{\mathsf{HS}} \varphi(a(\sigma), \dot{w})\} \\ &= \{(\tau, q) \,;\, a^{-1}(\tau) \in \mathsf{dom}\dot{z} \text{ and } a^{-1}(q) \Vdash_{\mathsf{HS}} \varphi(a^{-1}(\sigma), \dot{w})\} \\ &= \{(\tau, q) \,;\, \tau \in \mathsf{dom}\dot{z} \text{ and } q \Vdash_{\mathsf{HS}} \varphi(\sigma, \dot{w})\} = \dot{y} \end{aligned}$$

Thus $y \in V(G)^{\mathcal{F}}$ and separation holds.

Now let $x \in V(G)^{\mathcal{F}}$ and let $\dot{x} \in \mathsf{HS}$ be a name for x. For the union axiom take $\tau \stackrel{\text{def}}{=} \{(\sigma, \mathbb{1}) \ ; \exists \pi \in \mathsf{dom}\dot{x},\ \sigma \in \mathsf{dom}\pi\}$ and remember that if $a \in \mathsf{sym}\dot{x}$ then $a(\dot{x}) = \dot{x}$. This means that the names in $\mathsf{dom}\dot{x}$ may be permuted with each other but overall $\dot{x}$ stays the same; thus $a\tau = \tau$ as well. Clearly $\tau^G \supseteq \bigcup x$ holds and so because of separation we have that union also holds.

For pairing of $x, y \in V(G)^{\mathcal{F}}$ with hereditarily symmetric names $\dot{x}, \dot{y}$ respectively, we take $\tau \stackrel{\text{def}}{=} \{(\dot{x}, \mathbb{1}), (\dot{y}, \mathbb{1})\}$. For the powerset of x we take the name $\sigma \stackrel{\text{def}}{=} \{(\pi, 1) \ ; \pi \in \mathsf{HS}$ and $\mathsf{dom}\pi \subset \mathsf{dom}\dot{x}\}$. This σ is in HS and gives a superset of the powerset of x. So using separation we get that $V(G)^{\mathcal{F}}$ satisfies the powerset axiom.

Exercise 3. *Prove that $V(G)$ satisfies the axiom of replacement.*

qed

In the older method of permutation models it's common to build the normal filter via a normal ideal. The following notion of a symmetry generator corresponds to the notion of a normal ideal.

Definition 3.8. For an $E \subseteq \mathcal{P}$ we take its pointwise stabilizer group

$$\mathsf{fix}_{\mathcal{G}} E \stackrel{\text{def}}{=} \{a \in \mathcal{G} \ ; \forall p \in E, a(p) = p\}$$

which is the set of automorphisms that do not move the elements of E (they fix E).Usually we'll just write $\mathsf{fix}E$.

Call $I \subseteq \mathcal{P}(\mathbb{P})$ a $\mathcal{G}$-symmetry generator if it's closed under taking unions and if for all $a \in \mathcal{G}$ and $E \in I$, there is an $E' \in I$ such that $a\mathsf{fix}Ea^{-1} \supseteq \mathsf{fix}E'$.

Exercise 4. *If I is a $\mathcal{G}$-symmetry generator then the set $\{\mathsf{fix}E \ ; E \in I\}$ generates a normal filter $\mathcal{F}_I$ over $\mathcal{G}$.*

Definition 3.9. We say that a set $E \in I$ supports a name $\sigma \in \mathsf{HS}$ if $\mathsf{sym}\sigma \supseteq \mathsf{fix}E$.

Constructing symmetric models with symmetry generators also helps describe a very nice property of sets of ordinals.

4. The approximation lemma

We can have a lot of grip on what a symmetric model construction does to the sets of ordinals if we could describe these (wellorderable) sets in some inner ZFC model of $V(G)$. In particular to know them by knowing only an initial part of the forcing construction. This is exactly what the approximation property guarantees. Before we go on to say what this property is let's take a look at the symmetry generators we must use to describe it.

Definition 4.1. Let $\mathbb{P}$ be a partial order and $\mathcal{G}$ an automorphism group of $\mathbb{P}$. A symmetry generator I is called projectable for $\mathbb{P}, \mathcal{G}$ if for every $p \in \mathbb{P}$ and every $E \in I$, there is a $p^* \in E$ that is minimal (with respect to the partial order) and unique such that $p^* \geq p$. Call this $p^* = p \restriction^* E$ the projection of p to E.

We are going to use only projectable I's. They will be comprised by either initial segments of the partial order (like with the Lévy collapse) or by reasonable chunks of the partial order (like in the case of a product).

In our running example we take the symmetry generator $L \stackrel{\text{def}}{=} \{E_n \,;\, n \in \omega\}$.

It's easy to see that L is a projectable symmetry generator and that for an $n \in \omega$, $p \restriction^* E_n \stackrel{\text{def}}{=} p \cap (\omega \times n \times \omega_n)$ for each $p \in \mathbb{P}$.

Now we can describe the approximation property.

Definition 4.2. Let $\mathbb{P}$ be a partial order, $\mathcal{G}$ an automorphism group of $\mathbb{P}$, and I be a projectable symmetry generator for $\mathbb{P}$,$\mathcal{G}$. We say that the triple $(\mathbb{P}, \mathcal{G}, I)$ has the approximation property if for any formula φ with n free variables, any names $\sigma_1, \ldots, \sigma_n \in \mathsf{HS}$ all with support $E \in I$, and any $p \in \mathbb{P}$,

$$p \Vdash \varphi(\sigma_1, \ldots, \sigma_n) \text{ implies that } p \restriction^* E \Vdash \varphi(\sigma_1, \ldots, \sigma_n).$$

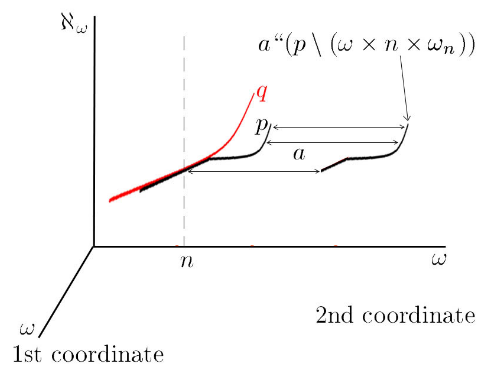
Next we present a way to guarantee the approximation property in our triples.

Lemma 1. *Let $\mathbb{P}$ be a partial order, $\mathcal{G}$ an automorphism group of $\mathbb{P}$, and I be a projectable symmetry generator for $\mathbb{P}, \mathcal{G}$. If for every $E \in I$, every $p \in \mathbb{P}$, and every $q \in \mathbb{P}$ such that $q \leq p \restriction^* E$, there is an automorphism*

$a \in \mathsf{fix}E$ such that $ap \parallel q$, then the triple $(\mathbb{P}, \mathcal{G}, I)$ has the approximation property.

Exercise 5. *Prove Lemma 1. (Hint: Use the properties of the forcing relation.)*

Let's see that in our running example the approximation lemma holds. For this we'll show that the triple $(\mathbb{F}, \mathcal{G}_{\mathbb{F}}, L)$ has the approximation property. Assume $E_n \in L$ and $q \in \mathbb{F}$ is such that $q \leq p\restriction^* E_n$.



Let $(\ )_2$ denote projection to the second coordinate. Since p is finite there is a set $A \subseteq \omega \setminus ((p)_2 \cup (q)_2 \cup n)$ that is equinumerous to $(p)_2 \setminus n$. Let $f : (p)_2 \setminus n \to A$ be a bijection and define a permutation a of ω by

$$a(m) \stackrel{\text{def}}{=} \begin{cases} m & \text{if } m \notin ((p)_2 \setminus n) \cup A \\ f(m) & \text{if } m \in (p)_2 \setminus n \\ f^{-1}(m) & \text{if } m \in A \end{cases}$$

This defines also an automorphism of $\mathbb{F}$ which we denote also by a. Clearly $a \in \mathsf{fix}E_n$ and $a(p) \parallel q$ so if G is an $\mathbb{F}$-generic filter, the approximation lemma holds for the symmetric model $V(G)^{\mathcal{F}_L}$.

The following is called the approximation lemma. It's a very useful way of describing sets of ordinals in symmetric models that have been constructed triples with the approximation property.

Lemma 4.3. If $\mathbb{P}$ is a partial order, $\mathcal{G}$ is an automorphism group, and I a projectable symmetry generator such that the triple $(\mathbb{P}, \mathcal{G}, I)$ has the approximation property then for any set of ordinals $X \in V(G)^{\mathcal{F}_I}$ there is an

$E \in I$ and an E-name for X, i.e., we can say that

$$X \in V[G \cap E].$$

Exercise 6. *Prove Lemma 4.3.*

5. More on the Feferman-Lévy model

With this section we give some facts about the Feferman-Lévy model $V(G)^{\mathcal{F}_L}$ and with them, an insight on how we work with symmetric models and the approximation lemma.

Theorem 5.1. The ordinal ω_ω^V is a cardinal in $V(G)^{\mathcal{F}_L}$.

Proof. Assume towards contradiction that g is a surjection from ω onto ω_ω^V that is in $V(G)^{\mathcal{F}_L}$ and let $\dot{g}$ be a symmetric name for g. By the Forcing Theorem (see [Jec03, Theorem 14.6]), there is a $p_0 \in G$ such that p_0 forces "$\dot{g}$ is a function from $\check{\omega}$ onto $\check{\omega}_\omega^V$". Since $g \in V(G)^{\mathcal{F}_L}$, $\mathsf{sym}(\dot{g}) \in \mathcal{F}_L$, i.e., there is an $n \in \omega$ such that $\mathsf{fix}E_n \subseteq \mathsf{sym}(\dot{g})$. Fix n.

For every $k \in \omega$, define

$$A_k \stackrel{\text{def}}{=} \{\alpha \in \omega_\omega^V \,;\, \exists p \in \mathbb{F}(p \supseteq p_0 \text{ and } p \Vdash \dot{g}(k) = \alpha)\}.$$

Note that the requirement $p_0 \subseteq p$ is just to make sure that g is still a surjection from ω to ω_ω^V (look at [Jec03, Theorem 14.7(i)(a)]). If for every $k \in \omega$, it was true that $|A_\kappa| \leq \omega_n^V$ then $\omega_\omega^V \leq \bigcup_{k\in\omega} \omega_n^V$ which is a contradiction in V. Therefore, for at least one $k \in \omega$, $|A_k| \geq \omega_{n+1}^V$. Fix this k.

For every $\alpha \in \omega_\omega^V$, define $B_\alpha = \{p \in \mathbb{F} \,;\, p \supset p_0 \text{ and } p \Vdash \dot{g}(k) = \alpha\}$.

Claim 1. Let $p, q \in \mathbb{F}$ such that $p, q \supseteq p_0$ and let α, $\beta \in \omega_\omega^V$. If $p \in B_\alpha$ and $q \in B_\beta$ and $\alpha \neq \beta$ then $q \perp p$.

Proof of Claim. Assume towards contradiction that $p \parallel q$, i.e., that there is an $r \supset p_0$ such that $r \leq p$ and $r \leq q$. For this r, since $p \in B_\alpha$ and by [Jec03, Theorem 14.7(i)(a)], $r \Vdash \dot{g}(k) = \alpha$. Similarly because $q \in B_\beta$, $r \Vdash \dot{g}(k) = \beta \neq \alpha$. Contradiction. qed claim

Therefore there must be at least ω_{n+1}^V incompatible conditions that force g to take different values with each condition. So define W to be this set of pairwise incompatible conditions such that for every $p \in W$, $p \supset p_0$. Also, there must be more than ω_{n+1}^V distinct ordinals α_p (one for every $p \in W$), such that for every $p \in W$, it holds that $p \Vdash \dot{g}(k) = \alpha_p$.

For every $p \in W$, by the approximation lemma we have that

$$p\restriction^* E_n \Vdash \dot{g}(k) = \alpha_p.$$

But note that the set $\{p\restriction^* E_n \ ; p \in \mathbb{F}\}$ has cardinality only ω_n^V, by definition of $\mathbb{F}$ and therefore this shows that there must be at least two distinct $p, q \in W$ such that $p\restriction^* E_n \Vdash \dot{g}(k) = \alpha_q$. Since $p \supseteq p\restriction^* E_n$, this means that $p \Vdash \dot{g}(k) = \alpha_q$. But by definition, $p \Vdash \dot{g}(k) = \alpha_p$ and we assumed $p \neq q$ to mean also that $\alpha_p \neq \alpha_q$. Then $\dot{g}$ cannot be a function so g cannot be a function. Contradiction. qed

The next lemma is an observation about the ordinals in $V(G)^{\mathcal{F}_L}$ compared to the ordinals in $V[G]$.

Lemma 5.2. The ordinal $\omega_1^{V(G)^{\mathcal{F}_L}}$ is singular in $V(G)^{\mathcal{F}_L}$ and for every $n \in \omega$, $\omega_{n+2}^{V(G)^{\mathcal{F}_L}} = \omega_{n+1}^{V[G]}$.

Proof. Since ω is absolute, $\omega^{V(G)^{\mathcal{F}_L}} = \omega^{V[G]}$. By Exercise 1, we have that in $V(G)^{\mathcal{F}_L}$, there is no function that makes ω_ω^V countable but there are functions f_n that make all infinite ordinals smaller than ω_ω^V countable. By Theorem 5.1 we have that ω_ω^V is not countable in $V(G)^{\mathcal{F}_L}$. This means firstly that ω_ω^V is a cardinal and in particular $\omega_\omega^V = \omega_1^{V(G)^{\mathcal{F}_L}}$. Secondly, we have that $(\mathsf{cf}(\omega_\omega^V))^V = \omega$. Since for models $V \subseteq V(G)^{\mathcal{F}_L}$, the cofinality can only decrease from V to $V(G)^{\mathcal{F}_L}$, we have that $\mathsf{cf}(\omega_1^{V(G)^{\mathcal{F}_L}}) \leq \mathsf{cf}(\omega_\omega^V) = \omega$ and therefore in $V(G)^{\mathcal{F}_L}$, ω_1 is singular.

Since $V[G] \models \mathsf{AC}$ (see [Jec03, Theorem 14.5]), it holds that countable unions of countable sets are countable in $V[G]$ and since $\omega_\omega^V = \bigcup_{n\in\omega} \omega_n^V$ holds, by Exercise 1 we get that ω_ω^V is countable in $V[G]$ and therefore cannot be $\omega_1^{V[G]}$. Note that the partial order $\mathbb{F}$ is a subset of the partial order $\mathsf{Fn}(\omega \times \omega, \omega_\omega)$ and by [Kun80, Lemma 6.10], $\mathsf{Fn}(\omega \times \omega, \omega_\omega)$ has the $(\omega_\omega^{<\omega})^+$-c.c. in V. We have that

$$(\omega_\omega^{<\omega})^+ = (\omega_\omega)^+ = \omega_{\omega+1}$$

and so in V the partial order $\mathsf{Fn}(\omega \times \omega)$ has the $\omega_{\omega+1}$-c.c.. By [Kun80, Lemma 6.9] and since ω is regular, this means that $\mathsf{Fn}(\omega \times \omega)$ preserves cardinals and cofinalities $\geq \omega_{\omega+1}^V$. So none of these ordinals above and with

$\omega^V_{\omega+1}$ has collapsed in $V[G]$ and therefore neither in $V(G)^{\mathcal{F}_L}$. So, $\omega_1^{V[G]} = \omega^V_{\omega+1} = \omega_2^{V(G)^{\mathcal{F}_L}}$ and for the same reasons, for every $n \in \omega$, $\omega_{n+2}^{V(G)^{\mathcal{F}_L}} = \omega_{n+1}^{V[G]}$ holds. qed

So indeed in the model $V(G)^{\mathcal{F}_L}$, $\omega_1^{V(G)^{\mathcal{F}_L}}$ is singular and moreover countable unions of countable sets of reals are not necessarily countable. Now we are going to see that $\mathbb{R}$ is such a countable union of countable sets.

Theorem 5.3. The set of all reals in the symmetric model $V(G)^{\mathcal{F}_L}$ is a countable union of countable sets.

Proof. Using AC in $\mathbb{V}$, we get a function $\dot{} : x \mapsto \dot{x}$ from $V(G)^{\mathcal{F}_L}$ to HS, such that $(\dot{x})_G = x$. If $x \in V(G)^{\mathcal{F}_L}$, then we know that $\dot{x} \in \mathsf{HS}$ therefore there is n such that $\mathsf{fix}E_n \subseteq \mathsf{sym}(\dot{x})$. Define $C_n \stackrel{\text{def}}{=} \{x \in \mathbb{R}^{V(G)^{\mathcal{F}_L}} \,;\, \mathsf{fix}E_n \subseteq \mathsf{sym}(\dot{x})\}$ and note that

$$\mathbb{R}^{V(G)^{\mathcal{F}_L}} = \bigcup_{n\in\omega} C_n.$$

Therefore if we prove that for every $n \in \omega$, C_n is countable, then we proved the theorem.

For $x \in \mathbb{R}^{V(G)^{\mathcal{F}_L}}$ we define a name $\ddot{x}$ such that

$$\ddot{x} \stackrel{\text{def}}{=} \{(\check{k}, p\restriction^* E_n) \,;\, p \Vdash \check{k} \in \dot{x}\}.$$

It's clear that $(\dot{x})_G = (\ddot{x})_G = x$. Define

$$C'_n \stackrel{\text{def}}{=} \{\ddot{x} \,;\, x \in C_n\} \subseteq V^{\mathbb{F}}$$

and note that $\ddot{}$ is an injection from C_n into C'_n. So if $V(G)^{\mathcal{F}_L} \models$ "C'_n is countable", then C_n will be countable and we will have proved the theorem.

Clearly, $C'_n \subseteq \mathcal{P}^{(k)}\omega^V_n$ for some finite[2] k. In $V \subseteq V(G)^{\mathcal{F}_L}$ there is a bijection $\mathcal{P}^{(k)}\omega^V_n \rightarrowtail\!\!\!\!\twoheadrightarrow \omega^V_{n+k}$ because GCH holds in V. Thus we can get an injection

$$C_n \rightarrowtail C'_n \rightarrowtail \mathcal{P}^{(k)}\omega^V_n \rightarrowtail\!\!\!\!\twoheadrightarrow \omega^V_{n+k}.$$

But in $V(G)^{\mathcal{F}_L}$, ω_{n+k} is countable, therefore C_n is countable in $V(G)^{\mathcal{F}_L}$. qed

[2]Probably $k = 6$.

6. Making a successor cardinal measurable

Under the axiom of choice, measurable cardinals are limit cardinals (they are in fact inaccessible). In this section we'll show that you need choice to prove this, by constructing a model where any desired successor may become measurable.

First we translate Jech's classic symmetric model construction where ω_1 is made measurable, from Boolean valued models entirely into partial orders. We make it into a passe-partout construction where $\mathsf{LC}(\kappa)$ is some large cardinal axiom and we generalise the construction so that we can make any given successor of a regular cardinal measurable, weakly compact, etc.. This way we'll be able to easily use this construction for many results. The original construction from Jech can be found in [Jec78, page 476] and [Jec68].

After we give the construction, we'll show a few immediate results.

6.1. **The basic construction.** Assume that we are in a model of ZFC in which there is an inaccessible cardinal κ and a regular cardinal $\eta < \kappa$. Note that here we can have $\eta = \omega$, or $\eta = \aleph_{\omega+1}$, etc.. Let

$$\mathbb{P} \stackrel{\text{def}}{=} \{p : \eta \rightharpoonup \kappa \ ; \ |p| < \eta \text{ and } p \text{ is injective}\},$$

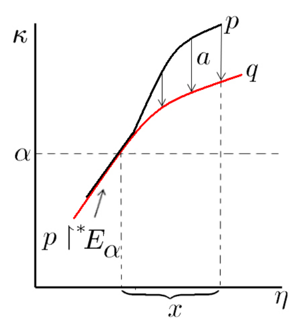
where $\rightharpoonup$ denotes a partial function. The ordering here is reverse inclusion, i.e., $p \leq q$ iff $p \supseteq q$. Let $\mathcal{G}$ be the full permutation group of κ and extend it to the partial order by permuting the range of the conditions, i.e., for $a \in \mathcal{G}$ and $p \in \mathbb{P}$, $a(p) \stackrel{\text{def}}{=} \{(\xi, a(\beta)) \ ; \ (\xi, \beta) \in p\}$.

Let I be the symmetry generator that is induced by the ordinals between η and κ, i.e., $I \stackrel{\text{def}}{=} \{E_\alpha \ ; \ \eta < \alpha < \kappa\}$, where $E_\alpha \stackrel{\text{def}}{=} \{p \cap (\eta \times \alpha) \ ; \ p \in \mathbb{P}\}$. It is clear that this is a projectable symmetry generator and for a $p \in \mathbb{P}$, $p\restriction^* E_\alpha = p \cap (\eta \times \alpha)$.

Take the symmetric model $V(G) = V(G)^{\mathcal{F}_I}$. for this model the approximation lemma holds.

Lemma 6.1. The triple $(\mathbb{P}, \mathcal{G}, I)$ has the approximation property, i.e., for every $\alpha \in (\eta, \kappa)$, every $p \in \mathbb{P}$ and every $q \in \mathbb{P}$ such that $q \leq p\restriction^* E_\alpha$, there is an automorphism $a \in \mathsf{fix}(E_\alpha)$ such that $a(p) \parallel q$.

Proof. Let $x \stackrel{\text{def}}{=} (\mathsf{dom} p \cap \mathsf{dom} q) \setminus \mathsf{dom}(p\restriction^* E_\alpha)$. Since the conditions are $< \kappa$-large there is a permutation a of κ that sends every $p(\xi)$ to $q(\xi)$ for $\xi \in x$ as the picture below shows. Clearly $a(p) \parallel q$. qed



In this model κ has become η^+.

Proposition 6.2. In $V(G)$, κ is the successor of η^+.

Proof. Let $\gamma < \kappa$ and define the following name.

$$\tau \stackrel{\text{def}}{=} \{(\check{p}, p) \; ; \; p \in \mathbb{P}\restriction^* E_\gamma\}.$$

This is a hereditarily symmetric $\mathbb{P}$-name (supported by γ) and τ_G is a bijection of η onto γ.

Assume now that there is some $\beta < \kappa$ and a bijection $f : \beta \to \kappa$ in $V(G)$. Let $\dot{f} \in \mathsf{HS}$ be a name for f with support δ. Then $f \in V[G \cap E_\delta]$ which is impossible since E_δ has cardinality $< \kappa$ and so it has the κ-cc. qed

Since $\mathbb{P}$ has the κ^+-cc, all cardinals above κ are also preserved. Moreover, since $\mathbb{P}$ is η-closed and all cardinals $\leq \eta$ are preserved as well.

Exercise 7. *In $V(G)$, the powerset $\mathcal{P}(\kappa)$ is not wellorderable, in particular, it's a κ-long union of κ-sized sets. Therefore AC_κ fails, but for all $x \in H_\kappa$, x is wellorderable.*

For our large cardinal considerations this lemma is optimal since according to Bull's [Bul78, Corollary 0.3], AC_κ cannot hold and κ be a successor cardinal that is measurable or weakly compact.

6.2. **Measurability.** Here we give Jech's result that the theory "ZF + ω_1 is measurable" is consistent relative to "ZFC + there exists a measurable". Our construction uses partial orders but one can see the Boolean valued model argumentation in [Jec78, page 476] and [Jec68].

By "κ is measurable" here we mean that there is a κ-complete ultrafilter U over κ. This is a set U such that U is an ultrafilter, i.e., it doesn't contain the emptyset, it's closed under taking supersets, it's closed under taking finite intersections, and for every subset $X \subseteq \kappa$, either X or it's complement $\kappa \setminus X$ are in U. Also, U is κ-complete, i.e., closed under taking less than κ-sized intersections.

Lemma 6.3. If κ is a measurable cardinal and $\eta < \kappa$ is a regular cardinal then there is a symmetric model in which η^+ is measurable.

Note that this lemma gives an infinity of consistency strength results, by replacing η with a description for a regular cardinal such as "η is ω_1", etc..

Proof. Take $V(G)$ to be Jech's construction with the same notation as in §6.1, and let U be a κ-complete ultrafilter over κ in V. In $V(G)$ define the set

$$W \stackrel{\text{def}}{=} \{x \subseteq \kappa\,;\, \exists y \in U, y \subseteq x\}.$$

Claim 2. In $V(G)$, W is an ultrafilter over κ.

Proof of Claim. Let $(X \subseteq \kappa)^{V(G)}$ and let $\dot{X} \in \mathsf{HS}$ be a canonical name for X with support E_γ for some $\gamma < \kappa$. Define

$$A \stackrel{\text{def}}{=} \{p{\restriction}^* E_\gamma \in \mathbb{P}\,;\, \text{there is an } \alpha < \kappa,\ p \Vdash \check{\alpha} \in \dot{X}\}.$$

Since $|E_\gamma| < \kappa$, $|A| < \kappa$. Define for every $x \subseteq A$ the set

$$Y_x \stackrel{\text{def}}{=} \{\alpha \in \kappa\,;\, \forall p \in x,\ p \Vdash \check{\alpha} \in \dot{X} \text{ and} \forall q \notin x,\ q \nVdash \check{\alpha} \in \dot{X}\}.$$

This $\{Y_x\,;\, x \subseteq A\}$ is a partition of κ and it has size less than κ because κ is inaccessible. So there is a unique $x \subseteq A$ such that Y_x is in U. Fix x. If there is $p \in x \cap G$, then $Y_x \subseteq X$ and if there is no $p \in x \cap G$ then $Y_x \subseteq \kappa \setminus X$. Hence either X or $\kappa \setminus X$ have a subset in U. qed claim

Exercise 8. *Similarly show that the set W is also κ-complete.*

Therefore κ which now is the η^+ of $V(G)$, is measurable in $V(G)$. qed

As noted by Lorenz Halbeisen, in Jech's model κ has also a normal measure.

Exercise 9. *In $V(G)$ there is a normal measure on κ.*

Question *Can there be a model of* ZF *with a measurable cardinal that has no normal measures?*

7. Making a successor cardinal weakly compact

This is a version of the Jech model, in Jech's [Jec68]. There the construction uses both partial orders and Boolean values and the proofs concern measurability. We are going to talk about weak compactness. Jech in his [Jec68] already notes this as a possibility with this model.

Definition 2. For cardinals κ, ξ, define

$$[\kappa]^\xi \stackrel{\text{def}}{=} \{x \subseteq \kappa \,;\, |x| = \xi\},$$

the set of ξ-sized subsets of κ.

For cardinals $\kappa, \lambda, \xi, \rho$, the partition relation

$$\kappa \to (\lambda)^\xi_\rho$$

means that for every partition $f : [\kappa]^\xi \to \rho$ there is a set $H \subseteq \kappa$ such that H has cardinality λ and it's homogeneous for f, i.e., $|f``[H]^\xi| = 1$.

A cardinal κ is weakly compact iff $\kappa \to (\kappa)^2_2$.

It follows from the axiom of choice that weakly compact cardinals are inaccessible, i.e., for every $\alpha < \kappa$, the powerset $\mathcal{P}(\alpha)$ has cardinality less that κ.

So from now on assume that we're in a model $V \models \mathsf{ZFC}$ + "there exists a weakly compact cardinal κ". Let η be a cardinal strictly below κ. We'll make a symmetric model where $\kappa = \eta^+$ and κ is still weakly compact.

Let

$$\mathbb{P} \stackrel{\text{def}}{=} \{p : \eta \rightharpoonup \kappa \,;\, |p| < \eta \text{ and } p \text{ is injective}\},$$

where $\rightharpoonup$ denotes a partial function. The ordering here is reverse inclusion, i.e., $p \leq q$ iff $p \supseteq q$. Let $\mathcal{G}$ be the full permutation group of κ and extend it to the partial order by permuting the range of the conditions, i.e., for $a \in \mathcal{G}$ and $p \in \mathbb{P}$,

$$a(p) \stackrel{\text{def}}{=} \{(\xi, a(\beta)) \; ; \; (\xi, \beta) \in p\}.$$

Denote also by $\mathcal{G}$ the set of the extended a.

Let I be the symmetry generator that is induced by the ordinals between η and κ, i.e., $I \stackrel{\text{def}}{=} \{E_\alpha \; ; \; \eta < \alpha < \kappa\}$, where

$$E_\alpha \stackrel{\text{def}}{=} \{p \cap (\eta \times \alpha) \; ; \; p \in \mathbb{P}\}.$$

It is clear that this is a projectable symmetry generator and for a $p \in \mathbb{P}$, $p{\restriction}^* E_\alpha = p \cap (\eta \times \alpha)$.

Take the symmetric model $V(G) = V(G)^{\mathcal{F}_I}$.

Exercise 10. *Prove that the approximation lemma holds for $V(G)$. That is, prove that the triple $(\mathbb{P}, \mathcal{G}, I)$ has the approximation property.*

Exercise 11. *Prove that in $V(G)$, every $\alpha \in (\eta, \kappa)$ is collapsed to η, i.e., that there is a hereditarily symmetric name $\tau_\alpha \in \mathsf{HS}$ such that if G is $\mathbb{P}$-generic, τ_α^G is a surjection from η to α. (Hint: For every such α define this τ_α.)*

Note that $\mathbb{P}$ has the κ^+-cc, so all cardinals above κ are preserved. Moreover, $\mathbb{P}$ is η-closed and and so all cardinals $\leq \eta$ are preserved as well. Show that κ is also preserved.

Exercise 12. *Prove that in $V(G)$, $\kappa = \eta^+$, i.e., prove that a collapsing function for κ cannot exist. (Hint: Use the approximation lemma.)*

Finally, such a model preserves several large cardinals. In our case:

Exercise 13. *Prove that in $V(G)$, κ is still weakly compact. (Hint: Try to define a homogeneous set by using the approximation lemma and by using the inaccessibility of κ. Note that by* [Kan03, Theorem 7.8]*, in V it holds that for every $n \in \omega$ and every $\lambda < \kappa$, $\kappa \to (\kappa)^n_\lambda$.)*

BONUS: Construct a symmetric model for which the approximation lemma holds.

References

[Bul78] Everett L. Bull, Jr. Consecutive large cardinals. *Annals of Mathematical Logic*, 15:161–191, 1978.

[Coh63] Paul J. Cohen. The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences of the USA*, 50:1143–1148, 1963.

[Coh64] Paul J. Cohen. The independence of the continuum hypothesis ii. *Proceedings of the National Academy of Sciences of the USA*, 51:105–110, 1964.

[FL63] Solovay Feferman and Azriel Lévy. Independence results in set theory by Cohen's method, ii (abstract). *Notices of the American Mathematical Society*, 10:593, 1963.

[Fra22] A. Abraham H. Fraenkel. Zu den Grundlagen der Cantor-Zermeloschen Mengelehre. *Mathematische Annalen*, 86:230–237, 1922.

[Jec68] Thomas J. Jech. ω_1 can be measurable. *Israel Journal of Mathematics*, 6:363–367, 1968.

[Jec71] Thomas J. Jech. *Lectures in set theory, with particular emphasis on the method of forcing*, volume 217 of *Lecture Notes in Mathematics*. Springer, 1971.

[Jec73] Thomas J. Jech. *The axiom of choice*. Dover publications, INC., 1973.

[Jec78] Thomas J. Jech. *Set theory*. New York Academic Press, 1st edition, 1978.

[Jec03] Thomas J. Jech. *Set theory*. Springer, 3rd edition, 2003.

[Kan03] Akihiro Kanamori. *The higher infinite*. Springer, 2nd edition, 2003.

[Kun80] Kenneth Kunen. *Set theory: an introduction to independence proofs*. Elsevier, 1980.

[Mos39] Andrzej Mostowski. Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip. *Fundamenta Mathematicae*, 32:201–252, 1939.

[Spe57] Ernst P. Specker. Zur Axiomatik der Mengenlehre (Fundierungs und Auswahlaxiom). *Zeitschift fur mathematische Logik und Grundlagen der Mathematik*, 3:173–210, 1957.