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Notes on the affine Hecke algebra and equivariant K-theory

1 Conventions and preliminaries

Let F be a non-archimedean local field with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}_q$. For a connected semisimple linear algebraic group A we shall write $W(A)$, $\tilde{W}(A)$ and $W_{\mathrm{aff}}(A)$ for the Weyl group, extended affine Weyl group, and affine Weyl group of A, respectively. We shall write $I(A)$ for the an Iwahori subgroup corresponding to a silently chosen Borel subgroup in $A(\mathbb{F}_q)$, and $\mathcal{F}\ell(A)$ for the corresponding affine flag variety $A(F)/I(A)$. Where there is no danger of confusion we will write A also for $A(F)$. Note however that we write $A^\vee$ for the Langlands dual of A, and as we consider only $\mathbb{C}$-points of $A^\vee$, we write $A^\vee = A^\vee(\mathbb{C})$.

Now fix G be a semisimple linear algebraic group and $\tilde{W} = W \ltimes X_*(T)$ its affine Weyl group. Let H be the affine Hecke algebra corresponding to $(\tilde{W}, S)$. It is frequently claimed that there is an isomorphism of $\mathbb{Z}[q, q^{-1}]$-algebras

$$H \to K_0^{G^\vee \times \mathbb{C}^\times}(\mathrm{St})$$

where St is the Steinberg variety of the complex Langlands dual group $G^\vee = G^\vee(\mathbb{C})$ of G. As known to all experts, this hypothesis can be dispensed with via passing to the universal covering group, but formally this doesn't seem to be done anywhere in writing, or even mentioned much of the time. This has tripped me up enough times that I decided to write up some pretty informal notes for myself.

2 The p-adic side

Let $(X^*, \Phi, X_*, \Phi^\vee)$ be the root datum for (G, T). Let $Q = \mathbb{Z}\Phi \subset X^*$ be the root lattice.

Definition 1. A group G is *simply-connected* if the coroot lattice is equal to the coweight lattice. The *fundamental group* of G is $\pi_1(G) := X^\vee/\mathbb{Z}\Phi^\vee = X_*(T)/\mathbb{Z}\Phi^\vee$ the coweight lattice modulo the coroot lattice.

Definition 2. A group $G^\vee$ is *of adjoint type* if it has trivial centre.

Langlands duality exchanges the centre and the fundamental group, so that

$$G \text{ is simply connected iff } G^\vee \text{ is of adjoint type}$$

Then we have the following diagram

$$\begin{array}{ccccccccc}
X & R & G & \pi_1(G) & X_* = \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Z}) & R^\vee & G^\vee & Z(G^\vee) & \mathrm{Hom}(Z(G^\vee),\mathbb{C}^\times) \\
\uparrow & {\scriptstyle \mathrm{id}}\uparrow & \downarrow{\scriptstyle \pi} & \downarrow & \downarrow & {\scriptstyle \mathrm{id}}\uparrow & {\scriptstyle p}\uparrow & \uparrow & \downarrow \\
Q & R & G^{\mathrm{ad}} & \pi_1(G^{\mathrm{ad}}) & \mathrm{Hom}_{\mathbb{Z}}(Q,\mathbb{Z}) & R^\vee & (G^{\mathrm{ad}})^\vee & \mathbb{Z}((G^{\mathrm{ad}}))^\vee & \mathrm{Hom}(Z((G^{\mathrm{ad}})^\vee),\mathbb{C}^\times).
\end{array} \tag{1}$$

and we see that $(G^{\mathrm{ad}})^\vee = \widetilde{G^\vee}$ is the universal cover of the dual group (in the sense of algebraic groups). Here $G^{\mathrm{ad}} = G/Z(G)$ is the *adjoint group* of G. Note that as we take Langlands dual groups over $\mathbb{C}$, the universal cover corresponds to that of Lie groups and we can deduce the surjections of centres that way.

The root data for these groups are as follows:

$$G: \ (X \supset Q, \Phi, X^\vee \supset Q^\vee, \Phi^\vee), \qquad G^\vee: \ (X^\vee \supset Q^\vee, \Phi^\vee, X \supset Q, \Phi)$$

$$G^{\mathrm{ad}} \ : (Q, \Phi, \mathrm{Hom}_{\mathbb{Z}}(Q,\mathbb{Z}), \Phi^\vee), \qquad (G^{\mathrm{ad}})^\vee \ : (\mathrm{Hom}_{\mathbb{Z}}(Q,\mathbb{Z}), \Phi^\vee, Q, \Phi).$$

Note in particular that $W_{\mathrm{aff}}(G) = W_{\mathrm{aff}}(G^{\mathrm{ad}})$.

2.1 Components of the affine flag variety

Note that by diagram (1) we have an inclusion of the coweight lattice of G into the coweight lattice of G^{ad} which induces an inclusion of extended affine Weyl groups

$$\tilde{W}(G) \hookrightarrow \tilde{W}(G^{\mathrm{ad}})$$

as well as the above-noted inclusion of fundamental groups.

As $Z(G(k)) \subset B(k)$, we have $\pi(I(G)) = I(G^{\mathrm{ad}})$, and obtain an induced injection

$$\begin{array}{ccc}
G & \twoheadrightarrow & \mathcal{F}\ell(G) := G/I(G) = \coprod_{\omega\in\pi_1(G)} \mathcal{F}\ell(G)^{\omega} = \coprod_{\omega\in\pi_1(G)} \bigcup_{w\in\omega W_{\mathrm{aff}(G)}} I(G)\dot{w}I(G) \\
\downarrow \pi & & \downarrow \\
G^{\mathrm{ad}} & \twoheadrightarrow & \mathcal{F}\ell(G^{\mathrm{ad}}) := G^{\mathrm{ad}}/I(G^{\mathrm{ad}}) = \coprod_{\omega\in\pi_1(G^{\mathrm{ad}})} \mathcal{F}\ell(G^{\mathrm{ad}})^{\omega} = \coprod_{\omega\in\pi_1(G^{\mathrm{ad}})} \bigcup_{w\in\omega W_{\mathrm{aff}(G)}} I(G^{\mathrm{ad}})\dot{w}I(G^{\mathrm{ad}}),
\end{array}$$

where if $w \in W \ltimes X_*/W \ltimes R^\vee$, then π restricts to an isomorphism

$$\pi\colon I(G)\dot{w}I(G) \xrightarrow{\sim} I(G^{\mathrm{ad}})\dot{w}I(G^{\mathrm{ad}})$$

where on the right-hand side $\dot{w}$ is the same representative as on the left-hand side, only considered as a representative of a coset in $W \ltimes \mathrm{Hom}_{\mathbb{Z}}(Q,\mathbb{Z})/W \ltimes R^\vee$.

In summary $\mathcal{F}\ell(G)$ is just a subset of the connected components of $\mathcal{F}\ell(G^{\mathrm{ad}})$. All the connected components are isomorphic.

2.1.1 Bruhat order on $\tilde{W}$

By definition of the Bruhat order on $\tilde{W}$, two elements of $\tilde{W}$ are comparable only if they are in the same W_{aff}-coset.

2.2 Inclusion of affine Hecke algebras

Therefore extension by zero provides an injection of $\mathbb{Z}[q,q^{-1}]$-algebras

$$C_c^\infty[I(G)\backslash G/I(G)] \hookrightarrow C_c^\infty[I(G^{\mathrm{ad}})\backslash G^{\mathrm{ad}}/I(G^{\mathrm{ad}})],$$

where we mean functions with values in $\mathbb{Z}[q,q^{-1}]$ (which of course is just $\mathbb{Z}[q^{-1}]$ in this case). Further, we have a direct sum decomposition, as vector spaces

$$\bigoplus_{\omega\in\pi_1(G)} \mathrm{span}_{\mathbb{Z}[q,q^{-1}]}\,\{T_w\}_{w\in\omega W_{\mathrm{aff}(G)}} \hookrightarrow \bigoplus_{\omega\in\pi_1(G^{\mathrm{ad}})} \mathrm{span}_{\mathbb{Z}[q,q^{-1}]}\,\{T_w\}_{w\in\omega W_{\mathrm{aff}(G)}}$$

compatible with the above injection of algebras.

The content of these notes is writing down carefully the fact that the isomorphism of $\mathbb{C}[q,q^{-1}]$-algebras that exists when the dual group is simply-connected restricts to an isomorphism of the two subalgebras, which will follow from it being compatible with this direct sum decomposition and the below direct sum decomposition as a linear map.

2.3 Algebraic alternative

The inclusion of extended affine Weyl groups together with the Berstein presentation (see below) gives a totally algebraic way to see the inclusion of Hecke algebras: Consider the restriction

$$\mathrm{res}\colon X_* = \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(Q,\mathbb{Z}).$$

Then we get

$$W \ltimes X_* \hookrightarrow W \ltimes \mathrm{Hom}_{\mathbb{Z}}(Q,\mathbb{Z}),$$

and a linear injection

$$H_{W \ltimes X_*} \to H_{W \ltimes \mathrm{Hom}_{\mathbb{Z}}(Q,\mathbb{Z})}.$$

sending

$$T_w \mapsto T_w, \quad w \in W$$

and

$$\theta_\lambda \mapsto \theta_{\mathrm{res}\lambda}, \quad \lambda \in X_*.$$

Note that as every simple root α lies in $Q = \mathbb{Z}\Phi$, the pairing between X_* and X restricts to the pairing between Q and $\mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$, and so the relations in the Bernstein presentation are preserved, yielding an injection of algebras.

3 The K-theory side

3.1 The Bernstein presentation

Aside from the usual Coxeter presentation of H, there is another presentation due to Bernstein (and Bernstein-Zelevinskii in type A) and first published by Lusztig that appears naturally on the K-theory side.

The Bernstein presentation uses the basis $\{T_w\theta_\lambda\}_{w\in W,\lambda\in X_*}$, where

- T_w is the usual basis element T_w from the Coxeter presentation.
- Writing $\lambda = \lambda_1 - \lambda_2$ for $\lambda_i \in X_*^+$ dominant coweights, set

$$\theta_\lambda = q^{\frac{-\ell(\lambda_1)+\ell(\lambda_2)}{2}} T_{\lambda_1} T_{\lambda_2}^{-1}$$

  where we view $\lambda_i \in \tilde{W}$ when taking lengths.

The elements $\theta_\lambda T_w$ are also a $\mathbb{Z}[q, q^{-1}]$- basis. The relations are as follows:

- For finite simple reflections, the quadratic relation $T_s^2 = (q-1)T_s + q$.
- $\theta_\lambda \theta_{\lambda'} = \theta_{\lambda+\lambda'}$.
- The Bernstein relation: for a finite simple root α we have

$$\theta_\lambda T_{s_\alpha} - T_{s_\alpha}\theta_{s_\alpha(\lambda)} = (q-1)\frac{\theta_\lambda - \theta_{s_\alpha(\lambda)}}{1-\theta_{-\alpha^\vee}} \quad \text{if} \quad \alpha^\vee \notin 2X_*.$$

- A more complicated version of the last relation if $\alpha^\vee \in 2X_*$.

Definition 3. The commutative subalgebra spanned by the θ_λ is the *Bernstein subalgebra.*

Remark 1. In particular the simpler form holds for any simply-connected root system.

Remark 2. The fraction is in fact always an element of H.

Example 1. For $G = \mathrm{SL}_2(F)$ and α is unique simple root, then

$$\alpha^\vee : \pi \mapsto \mathrm{diag}(\pi, \pi^{-1}),$$

so $\alpha^\vee \notin 2X_*$ and if $\lambda = \alpha^\vee$, the last relation reads

$$\theta_{\alpha^\vee} T_{s_\alpha} - T_{s_\alpha}\theta_{s_\alpha(\alpha^\vee)} = \theta_{\alpha^\vee} T_{s_\alpha} - T_{s_\alpha}\theta_{-\alpha^\vee} = (q-1)\frac{\theta_{\alpha^\vee} - \theta_{-\alpha^\vee}}{1-\theta_{-\alpha^\vee}} = (q-1)(1+\theta_\alpha).$$

Example 2. If $G = \mathrm{PGL}_2(F)$, an example that is not simply-connected, we have the following relation.

Recall that

$$2\rho = \alpha \colon \mathrm{GL}_2(\mathbb{C}) \ni \mathrm{diag}(a,b) \mapsto ab^{-1}$$

defines the usual simple root for SL_2, and descends to the quotient to define a weight of PGL_2.

This is not true of

$$\rho \colon \mathrm{GL}_2(\mathbb{C}) \ni \mathrm{diag}(a,b) \mapsto a.$$

This issue of $\mathbb{Z}$ versus $2\mathbb{Z}$ is good to keep in mind.

Let α be the root of SL_2 as above. The coroot lattice of of PGL_2 is $\mathbb{Z}\langle\alpha\rangle$. The coweight lattice of PGL_2 is $\mathbb{Z}\langle\frac{\alpha}{2}\rangle \supset \mathbb{Z}\langle\alpha\rangle$. In this case the last relation is of the form

$$\theta_{\frac{\alpha}{2}} T_{s_\alpha^\vee} - T_{s_\alpha^\vee} \theta_{-\frac{\alpha}{2}} = (q-1)\frac{\theta_{\frac{\alpha}{2}} - \theta_{-\frac{\alpha}{2}}}{1-\theta_{-\alpha}} = (q-1)\theta_{\frac{\alpha}{2}}$$

Compare this with the relation in K-theory to see why we are forced to take one side Langlands dual.

Proposition 1 ([2] corollary 5.7)**.** *We have* $\operatorname{supp}\theta_\lambda \subset \left\{x \in \tilde{W} \,\middle|\, x \leq \lambda\right\}$. *In particular,* $\operatorname{supp}\theta_\lambda \subset \mathcal{F}\ell^\lambda$.

3.1.1 Aside: Discrete-series representations

Definition 4. A representation belongs to the *discrete series* if all eigenvalues of the Bernstein subalgebra have absolute value strictly less than 1.

Example 3. For $G = \mathrm{SL}_2(F)$, there are four one-dimensional representations of H, defined in the Coxeter presentation by

1. $$\mathrm{St}\colon T_{s_0} \mapsto -1, \quad T_{s_1} \mapsto -1 \implies \theta_{\alpha^\vee} \mapsto q^{-1}.$$

2. $$\mathrm{triv}\colon T_{s_0} \mapsto q, \quad T_{s_1} \mapsto q \implies \theta_{\alpha^\vee} \mapsto q$$

3. $$T_{s_0} \mapsto -1, \quad T_{s_1} \mapsto q \implies \theta_{\alpha^\vee} \mapsto 1$$

4. $$T_{s_0} \mapsto q, \quad T_{s_1} \mapsto -1 \implies \theta_{\alpha^\vee} \mapsto 1$$

   The first representation is the *Steinberg representation.* It is discrete series for $q > 1$. The second is the *trivial representation.* It is not discrete series for $q > 1$ (it is fact "antitempered," or "antidiscrete series.") The last two representations are also not discrete series. They are however tempered, meaning that the eigenvalues of the Bernstein subalgebra are at most 1.

   The Steinberg and trivial representations deform the sign and trivial representations of the affine Weyl group $\langle s_0, s_1 \,|\, s_0^2 = s_1^2 = 1\rangle$, respectively.

3.2 Equivariant K-theory of the Steinberg variety

The basic reference is [1]

This section discusses objects on what we have chosen to be the Langlands dual side. Recall we write $G^\vee = G^\vee(\mathbb{C})$. Write St for the Steinberg variety of $G^\vee$, $K^{G^\vee\times\mathbb{C}^\times}(\mathrm{St}) = K_0^{G^\vee\times\mathbb{C}^\times}(\mathrm{St})$ for the equivariant K-theory of St. It is a $\mathbb{Z}[q,q^{-1}]$-algebra, where $\mathbb{Z}[q,q^{-1}] = K_0(\mathbf{Rep}(\mathbb{C}^\times))$ is the representation ring of $\mathbb{C}^\times$.

Inside Steinberg variety we have the smooth closed subvariety $\mathrm{St}_\Delta = \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \,|\, x \in \mathfrak{b}_1 = \mathfrak{b}_2 \text{ is nilpotent}\}$. It is isomorphic to the diagonal of $\tilde{\mathcal{N}}^\vee \times \tilde{\mathcal{N}}^\vee$.

Therefore we have the following diagram

$$
\begin{array}{ccccc}
(x, \mathfrak{b}, \mathfrak{b}) & \qquad & \mathrm{St}_\Delta & \xhookrightarrow{\Delta} & \tilde{\mathcal{N}}^\vee \times \tilde{\mathcal{N}}^\vee \\
\big\downarrow & & \big\downarrow{\scriptstyle \pi_\Delta} & & \big\downarrow{\scriptstyle \pi\times\pi} \\
\mathfrak{b} & & \mathcal{B}_\Delta & \xhookrightarrow{\Delta} & \mathcal{B}\times\mathcal{B}
\end{array}
$$

Let λ be a coweight of G, hence a weight of $G^\vee$. Recall the line bundle L_λ on $\mathcal{B}$, the flag variety of $G^\vee$. It is a smooth projective finite-dimensional variety.

The map π is $G^\vee$-equivariant, and so we get a $G^\vee \times \mathbb{C}^\times$-equivariant coherent sheaf $\mathcal{O}_\lambda := \pi_\Delta^* L_\lambda$ on St_Δ, which we view as a cohere sheaf on St supported on St_Δ.

Note that St_Δ is smooth, and so convolution in K-theory is the same as the tensor product in K-theory, which is defined. This makes multiplying the classes $[\mathcal{O}_\lambda]$ very simple, just like multiplying the θ_λ in the Bernstein presentation.

For $s \in S$, let $Y_s \subset \mathcal{B}\times\mathcal{B}$ be the corresponding $G^\vee$-orbit. Then $\bar{Y}_s = Y_s \sqcup \mathcal{B}_\Delta$ is smooth and

$$\bar{Y}_s \hookrightarrow \mathcal{B}\times\mathcal{B} \twoheadrightarrow \mathcal{B}$$

is a $\mathbb{P}^1$-fibration via the projection onto the first factor. Consider the sheaf $\Omega_{\bar{Y}_s/\mathcal{B}}$ of relative 1-forms. The conormal bundle $T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B})$ to $\bar{Y}_s$ is an irreducible component of St, and write $\mathcal{Q}_s = \pi_s^*\Omega_{\bar{Y}_s/\mathcal{B}^\vee}$.

3.3 Grading of equivariant K-theory of the Steinberg variety

The centre of $G^\vee$ acts trivially on St, and hence the equivariant K-theory acquires a $\mathrm{Hom}(Z(G^\vee), \mathbb{C}^\times)$-grading as a $\mathbb{Z}[q, q^{-1}]$-module. Thus we have a decomposition

$$K^{G^\vee\times\mathbb{C}^\times}(\mathrm{St}) = \bigoplus_{\chi\in\mathrm{Hom}(Z(G^\vee),U(1))} K^{G^\vee\times\mathbb{C}^\times}(\mathrm{St})_\chi$$

as a $\mathbb{Z}[q, q^{-1}]$-module.

Remark 3. This decomposition does not mean that there is a action of $Z(G^\vee)$ on K-theory, in the same way that in the case of a point, a group H does not act on its representation ring $R(H)$ or on $\mathbf{Rep}(H)$. What one can do given a subgroup K of H that acts on irreducible representations of H by a character, is partition the set of isomorphism classes of irreducible representations of H according to characters of K. The resulting direct sum decomposition of $R(A)$ as a $\mathbb{Z}$-module.

We have, as $G^\vee$ is semisimple,

$$\mathrm{Hom}(Z(G^\vee), \mathbb{G}_\mathrm{m}) \simeq X^\vee/Q^\vee = \pi_1(G),$$

by [3], theorem 4.22.

Note also that as $\widetilde{G^\vee} \to G^\vee$ is a finite cover of Lie groups (and $\widetilde{G^\vee}$ is the universal cover of $G^\vee$ as a Lie group (us having taken $\mathbb{C}$-points of a semisimple group)) $\mathrm{Lie}(G^\vee) = \mathrm{Lie}(\widetilde{G^\vee})$ and in particular the two groups have the same Steinberg varieties. Note further that $\ker(p)$, where p is the covering map from diagram (1), is a finite central subgroup of $\widetilde{(G^\vee)}$ and so a $G^\vee \times \mathbb{C}^\times$-equivariant coherent sheaf is a $\widetilde{(G^\vee)} \times \mathbb{C}^\times$-equivariant coherent sheaf with trivial $\ker(p)$-action on the fibres. Therefore we have an inclusion of $\mathbb{Z}[q, q^{-1}]$-algebras

$$K^{G^\vee\times\mathbb{C}^\times}(\mathrm{St}) = \bigoplus_{\chi\in\mathrm{Hom}(Z(G^\vee),U(1))} K^{G^\vee\times\mathbb{C}^\times}(\mathrm{St})_\chi \hookrightarrow K^{\widetilde{G^\vee}\times\mathbb{C}^\times}(\mathrm{St}) = \bigoplus_{\chi\in\mathrm{Hom}(Z(\widetilde{G^\vee}),U(1))} K^{\widetilde{G^\vee}\times\mathbb{C}^\times}(\mathrm{St})_\chi$$

compatible with the inclusion of duals of centres again from diagram (1), the point being that

$$K^{G^\vee\times\mathbb{C}^\times}(\mathrm{St})_\chi = K^{\widetilde{G^\vee}\times\mathbb{C}^\times}(\mathrm{St})_\chi$$

when χ is trivial on $\ker(p)$.

4 Checking the equivalence

Recall the usually-stated form of the isomorphism.

Theorem 1. *Let G be semisimple and $G^\vee$ be simply-connected. Then there is a $\mathbb{Z}[q,q^{-1}]$-algebra isomorphism*

$$\Theta\colon C_c^\infty[I(G)\backslash G/I(G)] \to K^{G^\vee \times \mathbb{C}^\times}(\mathrm{St}).$$

The isomorphism sends

$$\Theta\colon \theta_\lambda \mapsto [\mathcal{O}_{-\lambda}], \quad \lambda \in X_*$$

and

$$\Theta\colon T_s \mapsto -[q\mathcal{Q}_s] - [\mathcal{O}_0].$$

Remark 4. Before appearing in the book [1] as theorem 7.2.5, versions of various strength were published or announced by many authors. See [1] chapter 0 for a historical overview.

The whole claim is that the restriction of Θ lands where we claim it does.

Theorem 2. *Let G be semisimple. Then the following diagram of $\mathbb{Z}[q,q^{-1}]$-modules*

$$\begin{array}{ccc} C_c^\infty[I(G)\backslash G/I(G)] & \hookrightarrow & C_c^\infty[I(G^{\mathrm{ad}})\backslash G^{\mathrm{ad}}/I(G^{\mathrm{ad}})] \\ \downarrow{\scriptstyle \Theta\restriction_{C_c^\infty[I(G)\backslash G/I(G)]}} & & \downarrow{\scriptstyle\Theta} \\ K^{G^\vee\times\mathbb{C}^\times}(\mathrm{St}) & \hookrightarrow & K^{(G^{\mathrm{ad}})^\vee\times\mathbb{C}^\times}(\mathrm{St}) \end{array}$$

commutes.

Proof. First we will see that the elements T_s are mapped compatibly. Clearly they lie in the summand of $C_c^\infty(I(G)\backslash G/I(G))$ corresponding to $\omega = 1$. We must therefore show that the elements $\mathcal{Q}_s$ have trivial $Z(G^\vee)$ action as coherent sheaves. This is however clear, as $Z(G^\vee)$ is contained in any Borel subgroup $B^\vee$ of $G^\vee$, and hence acts trivially on $\mathcal{B}^\vee$ and $\mathcal{B}^\vee \times \mathcal{B}^\vee$, hence on functions on these varieties. Therefore the centre also trivially on the relative cotangent sheaves and on their pullbacks $\mathcal{Q}_s$ (see section 3.2 for the definition of $\mathcal{Q}_s$).

As a sanity check, recall that the line bundles L_λ for $\lambda \neq$ a weight of $G^\vee$ have sections which are *not* functions on $\mathcal{B}^\vee$ but rather functions on $G^\vee/N^\vee$ that transform under $T^\vee$ by λ if $B^\vee = T^\vee N^\vee$ and we view $\mathcal{B}^\vee = G^\vee/B^\vee$.

Recall that we know $\operatorname{supp}\theta_\lambda \subset \mathcal{F}\ell^\lambda$. We therefore want to show that $\mathcal{O}_{-\lambda} = \pi_\Delta^* L_{-\lambda}$ is scaled according to the weight λ of $Z(G^\vee)$.

Intuitively this is true by definition. If $r\colon G^\vee \twoheadrightarrow \mathcal{B}^\vee$, then sections of L_λ over an open subset $U \subset \mathcal{B}^\vee$ are functions $f\colon r^{-1}(U) \to \mathbb{C}$ such that $f(gb) = \lambda(b)^{-1} f(g)$, where λ is extended by inflation. As the centre is contained in any Borel, we have $f(zg) = f(gz) = \lambda^{-1}(z) f(g)$. Then as $G^\vee$ acts on $\mathcal{B}^\vee$ on the left, the action of $z \in Z^(G^\vee)$ on local sections of L_λ is

$$z\colon f \mapsto z\cdot f\colon x \mapsto f(z^{-1}x) = \lambda(z) f(x)$$

as required. Now the $G^\vee$-equivariant structure on $\mathcal{O}_\lambda$ is $\pi_\Delta^* I$, where

$$I\colon a^* L_\lambda \to p^* L_\lambda$$

is the isomorphism specifying the $G^\vee$-equivariant structure of L_λ as a sheaf on $\mathcal{B}^\vee$. Here a is the action map and p is projection as usual.

Upon restricting $\pi_\Delta^* I$ to $Z(G^\vee) \times \mathrm{St}_\Delta$ the action and projection maps are equal, and the $G^\vee$ -equivariant structure just provides an automorphism

$$\mathcal{O}_{Z(G^\vee)} \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{N}} \otimes_{\pi_\Delta^\bullet \mathcal{O}_{\mathcal{B}^\vee}} L_\lambda = (\mathcal{O}_{\tilde{N}} \otimes_{\pi_\Delta^\bullet \mathcal{O}_{\mathcal{B}^\vee}} L_\lambda)^{\oplus \# Z(G^\vee)} \to (\mathcal{O}_{\tilde{N}} \otimes_{\pi_\Delta^\bullet \mathcal{O}_{\mathcal{B}^\vee}} L_\lambda)^{\oplus \# Z(G^\vee)}.$$

The direct sum decomposition comes from the fact that the centre is finite discrete, and each component of the automorphism corresponds to the action of that element of the centre. Note that $Z(G^\vee)$ acts trivially on $\tilde{N}$, hence acts trivially on the first tensor factor. □

Corollary 1. *In theorem 1, the assumption that $G^\vee$ is simply-connected can be dropped.*

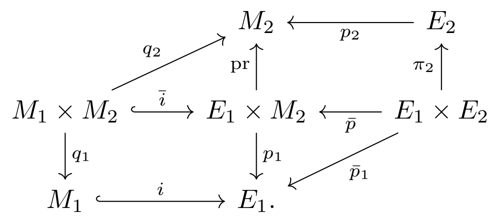
5 The asymptotic Hecke algebra and equivariant K-theory

The following is Lemma 5.4.27 in [1], although there are some inconsistencies of notation in the proof as typeset there.

Lemma 1. *We have the following commutative diagram of $R(G^\vee)$-modules:*

$$\begin{array}{ccc} K^G(Z) & \xrightarrow{\rho_E} & \mathrm{Hom}_{R(G)}(K^G(E_2), K^G(E_1)) \\ \downarrow{\scriptstyle \bar{i}^*\bar{p}_*} & & \downarrow{\scriptstyle \mathrm{Th}} \\ K^G(M_1 \times M_2) & \xrightarrow{\rho_M} & \mathrm{Hom}_{R(G)}(K^G(M_2), K^G(M_1)). \end{array}$$

Proof. It is helpful to draw the expanded diagram



Here the maps with indices are projections from products onto the indexed factor, except for $p_2\colon E_2 \to M_2$ which is just the bundle projection. The map pr is also projection onto the factor M_2.

We must prove that

$$(\bar{i}^*\bar{p}_*\mathcal{F}) \star \mathcal{G} \simeq i_1^*(\mathcal{F} \star \pi_2^* p_2^* \mathcal{G});$$

the right-hand side is the meaning of the map Th of hom-spaces induced by the Thom isomorphism.

Expanding, we need to prove

$$q_{1*}((\bar{i}^* p_*\mathcal{F}) \otimes^L q_2^*\mathcal{G}) \simeq i_1^*(\bar{p}_{1*}(\mathcal{F} \otimes^L \bar{p}^*\mathrm{pr}^*\mathcal{G})). \tag{2}$$

The lower square of the helpful diagram is Cartesian, and so we have

$$i_1^* p_{1*} = q_{1*}\bar{i}^* \implies i_1^* p_{1*}\bar{p}_* = i_1^*\bar{p}_1 = q_{1*}\bar{i}^*\bar{p}_*$$

by base-change and the commutativity of the lower triangle.

Therefore we can rewrite the RHS of (2) as

$$\begin{aligned} i_1^*(\bar{p}_2(\mathcal{F} \otimes^L \bar{p}\mathrm{pr}^*\mathcal{G})) &= q_{1*}\bar{i}^*\bar{p}_*(\mathcal{F} \otimes^L \bar{p}^*\mathrm{pr}^*\mathcal{G}) && (3) \\ &= q_{1*}\bar{i}^*(\bar{p}_*\mathcal{F} \otimes^L \mathrm{pr}^*\mathcal{G}) && (4) \\ &= q_{1*}(\bar{i}^*\bar{p}_*\mathcal{F} \otimes^L \bar{i}^*\mathrm{pr}^*\mathcal{G}) && (5) \\ &= q_{1*}(\bar{i}^*\bar{p}_*\mathcal{F} \otimes^L q_2^*\mathcal{G}) && (6) \\ & && (7) \end{aligned}$$

where in line 4 we used the projection formula. The projection formula is stated only in a limited way in [1], but the point here is that we can replace coherent sheaves by its finite locally-free resolution because we are working with smooth varieties everywhere. □

If $E_1 = E_2 = E$ and $M_1 = M_2 = M$ and $Z \subset E \times E$, then $K^G(Z)$ acquires the structure of a convolution algebra.

Then $\mathrm{Hom}_{R(G)}(K^G(E), K^G(E))$ and $\mathrm{Hom}_{R(G)}(K^G(M), K^G(M))$ become algebras as well, and $K^G(M \times M)$ is always an algebra under convolution, and we can ask if the last lemma upgrades to give a commutative diagram of $R(G)$.

Lemma 2. *We have the following commutative diagram of $R(G^\vee)$-algebras. This is Corollary 5.4.34 in* [1].

$$\begin{array}{ccc} K^G(Z) & \xrightarrow{\rho_E} & \mathrm{Hom}_{R(G)}(K^G(E), K^G(E)) \\ \Big\downarrow{\scriptstyle \bar{i}^* \bar{p}_*} & & \Big\downarrow{\scriptstyle \mathrm{Th}} \\ K^G(M \times M) & \xrightarrow{\rho_M} & \mathrm{Hom}_{R(G)}(K^G(M), K^G(M)). \end{array}$$

Remark 5. Note that if we know that ρ_M is injective, this is automatic. In particular it applies whenever the Künneth theorem from [1], §5.6, is known to hold for M. Note also that if $q_{1*}\mathcal{F} \neq 0$, then taking $\mathcal{G}$ to be a skyscraper on M, we see that $\rho_M(\mathcal{F})(\mathcal{G}) = q_{1^*}(\mathcal{F}) \neq 0$, so ρ_M is injective on such sheaves. It does however not seem that this pushforward is in general injective.

Proof. The map $\mathrm{Th}\colon \varphi \mapsto i^* \circ \varphi \circ (i^*)^{-1}$, and we obviously have $\mathrm{Th}(\varphi \circ \psi) = \mathrm{Th}(\varphi) \circ \mathrm{Th}(\psi)$, as required. □

References

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