Seminar on perverse sheaves

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April 3, 2024

Organizational meeting: January 26, 13:00 in N0.007.

Seminar Meeting time: Monday 10:00-12:00 in 1.007

Prerequisites for success: Basic algebraic geometry: Sheaves of vector spaces on complex varieties. Basic algebraic topology: Singular cohomology, *e.g.* of real and complex projective spaces. Some experience with derived categories and functors will be useful.

Requisites for success: Email me at `dawydiak@math.uni-bonn.de` to arrange a meeting the week before your talk. During this meeting we can discuss questions you have about the material and how best to present it.

Remember, the goal is for everyone to learn the material! In particular, most of your participation in the seminar will be as an audience member, so it is in everyone's interest for the talks to be understandable. Therefore, **plan your talk to take 75 minutes without questions, so that it can take 90 minutes with questions**.

Some of the talks include instructions to meet with the speaker of the next talk. The main point of these meetings is to ensure that the sum of your talks covers the stated material.

It is also good idea to rehearse your talk in front of a friend. This can be combined with meeting to discuss with the speaker of the next talk.

Please submit lecture notes (that is, what you plan to write on the board during your talk) for your talk to `dawydiak@math.uni-bonn.de`, preferably before but certainly no later than, the day of your talk. Scans of neatly handwritten notes are fine.

Postrequisites for success: According to the schedule, we will see relatively few applications, but after chapters 4, 5, 6, 9 of [1], it becomes possible to engage with many applications. After the seminar, you should be positioned to read these chapters.

Plan of talks. Detailed syllabi for talks 1–9 have been sent to speakers by email. Please let me know if you have not received yours. Syllabi for talks 10–14 will be sent to each speaker by email soon.

1. **Derived categories. April 8.**

   Derived categories and derived functors, following §A.5–A.6 of [1] and partly [3]. Main example: the derived category of sheaves of $\mathbb{C}$-vector spaces on a topological space.

   This meeting will be held at the MPIM (Vivatsgasse 7) in the seminar room. You must sign in as a guest at the reception desk when you enter the MPIM.

2. **Sheaves I. April 15.**

   Briefly recall our setting of sheaves of $\mathbb{C}$-vector spaces on a complex algebraic variety X with the analytic (metric) topology, and the (derived versions of the) familiar adjoint functors f_*, f_*. Follow §1.2–1.4

of [1]. Introduce the new functors $f_!$, $\otimes$, $\mathcal{R}Hom$, their appearance in the open-closed distinguished triangles, and proper base-change.

Talk to the speaker of the next talk.

3. **Sheaves II. April 22.**

   Local systems and representations of $\pi_1(X)$. Construction of $f^!$ as right adjoint to $f_!$. Proper base change. Follow §1.5–1.7 and part of §1.2 of [1].

   Talk to the speaker of the next talk.

4. **Towards Constructible sheaves I. April 29.**

   Define smooth morphisms as in [2], prove smooth base-change (this entails proving several theorems from §1.9). Prove base-change for locally-trivial fibrations.

   Talk to the speaker of the next talk.

5. **Towards Constructible sheaves II. May 6.**

   !-pullback for smooth morphisms, !-pullback of local systems to locally-closed inclusions.

6. **Constructible sheaves I. May 13**

   Stratifications. Examples: The cell decomposition of $\mathbb{CP}^n$, the cell decomposition of the Grassmannian as stratifications. The Schubert stratification of a flag variety. Filtrations by smooth varieties. The constructible derived category $D^b_c(X)$. The functor f^* preserves constructibility.

   Talk to the speaker for the next talk.

7. **Constructible sheaves II. May 27.**

   Nagata compactification, Ehresmann fibration theorem. Transverse locally trivial fibrations, Theorem 2.1.21 of [1]. The functors f_*, $f_!$, and $R\mathcal{H}om$ preserve constructibility.

   Talk to the speaker for the next talk.

   This meeting will be held virtually. Zoom Meeting ID: 612 949 9780. Password: The number of our usual meeting room.

8. **Constructible sheaves III. June 3.**

   Prove that Verdier duality is an involution, its interactions with all the pushforward and pullback functors. Prove that $f^!$ preserves constructibility. Compatibility of everything with $\boxtimes$.

   Talk to the speaker for the next talk.

9. **Triangulated categories. June 10**

   Triangulated categories, t-structures and their hearts, cohomology. Follow §A.4 and A.7 of [1].

10. **Perverse sheaves I. June 17**

    The perverse t-structure, t-exactness of $\mathbb{D}$, $-\otimes\mathcal{L}$, and $\boxtimes$. Emphasize the "lower triangularity" condition of perversity, reproduce the illustration from the end of §7.5 of [5].

    Talk to the speaker for the next talk.

11. **Perverse sheaves II. June 24.**

    Intersection cohomology "sheaves" (*i.e.* complexes, that are perverse), §3.3.

    Talk to the speaker for the next talk

12. **Perverse sheaves III. July 1.**

    Further examples of IC sheaves from [1], §3.10, IC sheaves on curves following [5].

13. **Perverse sheaves IV. July 8.**

    Affine pushforward and smooth pullback are t-exact.

14. **Perverse sheaves V. July 15.**

    Semismall morphisms and the decomposition theorem. Application: The Springer sheaf, following §8.1, 8.2 of [1].

References

[1] P. Achar, *Perverse Sheaves and Applications to Representation Theory*, Mathematical Surveys and Monographs, no. 258, American Mathematical Society, Providence, RI, 2021.

[2] D. Arapura, *Algebraic geometry over the complex numbers*, Springer, New York, 2012.

[3] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der mathematischen Wissenschaften, vol. 292, Springer-Verlag, Berlin-Heidelberg, 1990.

[4] R. Hotta, K. Takeuchi, and T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, Boston, 2008.

[5] G. Williamson, *Illustrated Guide to Perverse Sheaves*, available at `https://people.mpim-bonn.mpg.de/geordie/perverse_course/lectures.pdf`.