t-STRUCTURES

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ABSTRACT. These notes describe the machinery called t-structures for extracting an abelian category from a triangulated category.

CONTENTS

Let $\mathscr{T}$ be a triangulated category throughout these notes.

1. TRUNCATING t-STRUCTURES

Definition 1.1. A subcategory $\mathscr{B}$ of a category $\mathscr{A}$ is **isomorphism closed** (or **replete**) if every $\mathscr{A}$-isomorphism $h : A \to B$ with $B \in \mathscr{B}$ belongs to $\mathscr{B}$, which in particular implies that A, h, and h^{-1} belong to $\mathscr{B}$ as well. If in addition $\mathscr{B}$ is a full subcategory, then it is called **strictly full**.

Remark 1.2. In the case of full subcategories, it is sufficient to check that every $\mathscr{A}$-object that is isomorphic to a $\mathscr{B}$-object is also a $\mathscr{B}$-object.

Example 1.3. Consider an arbitrary topological property (i.e. a property of topological spaces that is preserved by homeomorphisms), then it determines a strictly full subcategory of Top.

Definition 1.4. Let $\mathscr{T}^{\leq 0}$ and $\mathscr{T}^{\geq 0}$ be (nonempty) strictly full subcategories of $\mathscr{T}$. For $n \in \mathbb{Z}$, let

$$\mathscr{T}^{\leq n} = \mathscr{T}^{\leq 0}[-n] \text{ and } \mathscr{T}^{\geq n} = \mathscr{T}^{\geq 0}[-n].$$

The pair $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ is a t**-structure** on $\mathscr{T}$ if the following three axioms hold:

(T1) We have $\mathscr{T}^{\leq -1} \subset \mathscr{T}^{\leq 0}$ and $\mathscr{T}^{\geq 1} \subset \mathscr{T}^{\geq 0}$.

(T2) If $X \in \mathscr{T}^{\leq 0}$ and $Y \in \mathscr{T}^{\geq 1}$, then $\mathrm{Hom}_{\mathscr{T}}(X, Y) = 0$.

(T3) For every $X \in \mathscr{T}$, there exists a distinguished triangle

$$X_0 \to X \to X_1 \to X_0[1]$$

in $\mathscr{T}$ with $X_0 \in \mathscr{T}^{\leq 0}$ and $X_1 \in \mathscr{T}^{\geq 1}$.

A t-structure is **bounded below** (resp. **bounded above**) if, for every $X \in \mathscr{T}$, there is an integer n such that $X \in \mathscr{T}^{\geq n}$ (resp. $X \in \mathscr{T}^{\leq n}$). It is **bounded** if it is bounded both below and above.

Proposition/Definition 1.5. *If $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ is a t-structure on $\mathscr{T}$, then so is $(\mathscr{T}^{\leq n}, \mathscr{T}^{\geq n})$ which is called the* **shifted t-structure**.

Proof. This is an immediate consequence of the shift functor $[1] : \mathscr{T} \xrightarrow{\sim} \mathscr{T}$. □

Proposition/Definition 1.6. *For an abelian category $\mathscr{A}$, the full subcategories of $D(\mathscr{A})$ given by*

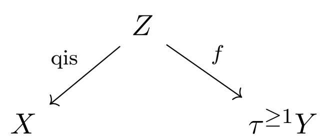
$$D(\mathscr{A})^{\leq 0} := \left\{X \in D(\mathscr{A}) : H^i(X) = 0 \text{ for } i > 0\right\}$$
$$\text{and } D(\mathscr{A})^{\geq 0} := \left\{X \in D(\mathscr{A}) : H^i(X) = 0 \text{ for } i < 0\right\}.$$

form a t-structure on $D(\mathscr{A})$, called the **natural t-structure**.

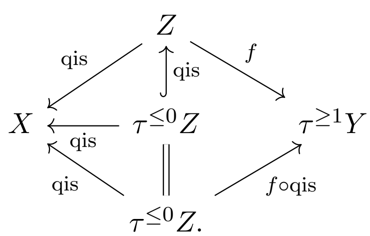
Proof. The axiom (T1) is clear. For (T2), let $X \in D(\mathscr{A})^{\leq 0}$ and $Y \in D(\mathscr{A})^{\geq 1}$ be arbitrary. The canonical morphism $Y \to \tau^{\geq 1}Y$ in $D(\mathscr{A})$ is an isomorphism, so we obtain a canonical identification

$$\operatorname{Hom}_{D(\mathscr{A})}(X, Y) = \operatorname{Hom}_{D(\mathscr{A})}(X, \tau^{\geq 1}Y).$$

Let



be an arbitrary morphism in $\operatorname{Hom}_{D(\mathscr{A})}(X, \tau^{\geq 1}Y)$, and consider the following commutative diagram



Since the morphism $f \circ \text{qis} : \tau^{\leq 0}Z \to \tau^{\geq 1}Y$ in $K(\mathscr{A})$ can only be represented by the zero morphism in $C(\mathscr{A})$, we obtain $\operatorname{Hom}_{D(\mathscr{A})}(X, \tau^{\geq 1}Y) = 0$. For (T3), one observes the distinguished triangle

$$\tau^{\leq 0}X \longrightarrow X \longrightarrow \tau^{\geq 1}X \longrightarrow (\tau^{\leq 0}X)[1]$$

in $D(\mathscr{A})$. It comes from for every $n \in \mathbb{Z}$ the short exact sequence

$$0 \longrightarrow \tau^{\leq n}X \longrightarrow X \longrightarrow X/\tau^{\leq n}X \longrightarrow 0$$

in $C(\mathscr{A})$. There is an associated distinguished triangle

$$\tau^{\leq n}X \longrightarrow X \longrightarrow X/\tau^{\leq n}X \longrightarrow (\tau^{\leq n}X)[1],$$

in $D(\mathscr{A})$. We then conclude using the canonical isomorphism $X/\tau^{\leq n}X \xrightarrow{\sim} \tau^{\geq n+1}$ in $D(\mathscr{A})$. □

Given a t-structure on $\mathscr{T}$, there follows two powerful adjoint functors:

Theorem 1.7 (Truncating t-structures)**.** *Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$.*

(i) *The inclusion* $\mathscr{T}^{\leq n} \hookrightarrow \mathscr{T}$ *admits a right adjoint* ${}^t\tau^{\leq n} : \mathscr{T} \to \mathscr{T}^{\leq n}$.

(ii) *The inclusion* $\mathscr{T}^{\geq n} \hookrightarrow \mathscr{T}$ *admits a left adjoint* ${}^t\tau^{\geq n} : \mathscr{T} \to \mathscr{T}^{\geq n}$.

(iii) *There exists a unique natural transformation* $\delta : {}^t\tau^{\geq n+1} \to [1]\,{}^t\tau^{\leq n}$ *such that, for every* $X \in \mathscr{T}$, *the diagram*

$$ {}^t\tau^{\leq n}X \longrightarrow X \longrightarrow {}^t\tau^{\geq n+1}X \xrightarrow{\delta_X} ({}^t\tau^{\leq n}X)[1] $$

is a distinguished triangle. Moreover, every distinguished triangle $A \to X \to B \to A[1]$ *with* $A \in \mathscr{T}^{\leq n}$ *and* $B \in \mathscr{T}^{\geq n+1}$ *is canonically isomorphic to this one.*

Proof. We first assume $n = 0$.

For an arbitrary $X \in \mathscr{T}$, pick a distinguished triangle

$$ X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1] $$

in $\mathscr{T}$ with $X_0 \in \mathscr{T}^{\leq 0}$ and $X_1 \in \mathscr{T}^{\geq 1}$. Put

$$ {}^t\tau^{\leq 0}X = X_0 \text{ and } {}^t\tau^{\geq 1}X = X_1. $$

Let $Y \in \mathscr{T}^{\leq 0}$ be arbitrary, and consider the exact sequence

$$ \mathrm{Hom}_{\mathscr{T}}(Y, X_1[-1]) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(Y, X_0) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(Y, X) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(Y, X_1). $$

We have $\mathrm{Hom}_{\mathscr{T}}(Y, X_1) = 0$ since $Y \in \mathscr{T}^{\leq 0}$ and $X_1 \in \mathscr{T}^{\geq 1}$. Similarly, $\mathrm{Hom}_{\mathscr{T}}(Y, X_1[-1]) = 0$ since $Y \in \mathscr{T}^{\leq 0}$ and $X_1[-1] \in \mathscr{T}^{\geq 2} \subset \mathscr{T}^{\geq 1}$. This implies a canonical identification

$$ \mathrm{Hom}_{\mathscr{T}}(Y, X_0) = \mathrm{Hom}_{\mathscr{T}}(Y, X). $$

Take ${}^t\tau^{\leq 0}X \to X$ to be $X_0 \to X$ of the distinguished triangle mentioned above.

Now we define ${}^t\tau^{\leq 0}$ on morphisms. Given a morphism $f : X \to Y$ in $\mathscr{T}$, let ${}^t\tau^{\leq 0} : {}^t\tau^{\leq 0}X \to {}^t\tau^{\leq 0}Y$ be the unique preimage of the composition ${}^t\tau^{\leq 0}X \to X \xrightarrow{f} Y$ in $\mathrm{Hom}_{\mathscr{T}}({}^t\tau^{\leq 0}X, Y)$ via the canonical identification

$$ \mathrm{Hom}_{\mathscr{T}}({}^t\tau^{\leq 0}X, {}^t\tau^{\leq 0}Y) = \mathrm{Hom}_{\mathscr{T}}({}^t\tau^{\leq 0}X, Y) $$

with ${}^t\tau^{\leq 0}X, {}^t\tau^{\leq 0}Y, Y$ replacing Y, X_0, X respectively in the previous identification. We then verify ${}^t\tau^{\leq 0}(gf) = {}^t\tau^{\leq 0}g \circ {}^t\tau^{\leq 0}f$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{T}$. Consider distinguished triangles

$$ X_0 \xrightarrow{c_X} X \longrightarrow X_1 \longrightarrow X_0[1], $$

$$ Y_0 \xrightarrow{c_Y} Y \longrightarrow Y_1 \longrightarrow Y_0[1], $$

$$ \text{and } Z_0 \xrightarrow{c_Z} Z \longrightarrow Z_1 \longrightarrow Z_0[1]. $$

Then ${}^t\tau^{\leq 0}f$ and ${}^t\tau^{\leq 0}g$ are uniquely given by the identities

$$ c_Y \circ {}^t\tau^{\leq 0}f = f \circ c_X \text{ and } c_Z \circ {}^t\tau^{\leq 0}g = g \circ c_Y $$

respectively. It follows that

$$ c_Z \circ ({}^t\tau^{\leq 0}g \circ {}^t\tau^{\leq 0}f) = (c_Z \circ {}^t\tau^{\leq 0}g) \circ {}^t\tau^{\leq 0}f = g \circ (c_Y \circ {}^t\tau^{\leq 0}f) = (g \circ f) \circ c_X = c_Z \circ {}^t\tau^{\leq 0}(gf), $$

which implies ${}^t\tau^{\leq 0}(gf) = {}^t\tau^{\leq 0}g \circ {}^t\tau^{\leq 0}f$. For $\mathrm{id}_X : X \to X$ in $\mathscr{T}$, we have

$$ c_X \circ {}^t\tau^{\leq 0}\mathrm{id}_X = \mathrm{id}_X \circ c_X = c_X = c_X \circ \mathrm{id}_X. $$

This yields ${}^t\tau^{\leq 0}\mathrm{id}_X = \mathrm{id}_X$. The naturality of the adjunction is clear from its construction.

Analogous considerations apply to ${}^t\tau^{\geq 1}$. Take $\delta : {}^t\tau^{\geq 1}X \to ({}^t\tau^{\leq 0}X)[1]$ to be $X_1 \to X_0[1]$ of the distinguished triangle mentioned above. Its functoriality follows from the commutative diagram

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{c_X} & X & \longrightarrow & X_1 & \longrightarrow & X_0[1] \\
\Big\downarrow {\scriptstyle {}^t\tau^{\leq 0}f} & & \Big\downarrow {\scriptstyle f} & & \Big\downarrow {\scriptstyle {}^t\tau^{\geq 1}f} & & \Big\downarrow {\scriptstyle ({}^t\tau^{\leq 0}f)[1]} \\
Y_0 & \xrightarrow{c_Y} & Y & \longrightarrow & Y_1 & \longrightarrow & Y_0[1].
\end{array}$$

Its uniqueness follows from $\mathrm{Hom}_{\mathscr{T}}(X_0, Y_1[-1]) = 0$ as $X_0 \in \mathscr{T}^{\leq 0}$ and $Y_1[-1] \in \mathscr{T}^{\geq 2} \subset \mathscr{T}^{\geq 1}$.

We have $\mathrm{Hom}_{\mathscr{T}}(A, X_1) = 0$ and $\mathrm{Hom}_{\mathscr{T}}(X_0, B) = 0$ since $A, X_0 \in \mathscr{T}^{\leq 0}$ and $X_1, B \in \mathscr{T}^{\geq 1}$. So there are morphisms of distinguished triangles indicated in the following commutative diagram

$$\begin{array}{ccccccc}
A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & A[1] \\
\Big\downarrow & & \Big\| & & \Big\downarrow & & \Big\downarrow \\
X_0 & \longrightarrow & X & \longrightarrow & X_1 & \longrightarrow & X_0[1] \\
\Big\downarrow & & \Big\| & & \Big\downarrow & & \Big\downarrow \\
A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & A[1].
\end{array}$$

Since $\mathrm{Hom}_{\mathscr{T}}(A, B[-1]) = 0$ as $A \in \mathscr{T}^{\leq 0}$ and $B[-1] \in \mathscr{T}^{\geq 2} \subset \mathscr{T}^{\geq 1}$, the composition $A \to X_0 \to A$ is uniquely id_A. Similarly, the composition $X_0 \to A \to X_0$ is uniquely id_{X_0}. Applying five lemma for triangulated categories, the same holds for B and X_1. This verifies the canonical isomorphism between the above two distinguished triangles where the latter is actually

$$ {}^t\tau^{\leq n}X \longrightarrow X \longrightarrow {}^t\tau^{\geq n+1}X \xrightarrow{\delta_X} ({}^t\tau^{\leq n}X)[1]. $$

By Proposition/Definition 1.5, we have

$$ {}^t\tau^{\leq n}X = {}^t\tau^{\leq 0}(X[n])[-n] $$
$$ \text{and } {}^t\tau^{\geq n+1}X = {}^t\tau^{\geq 1}(X[n])[-n] $$

for arbitrary $n \in \mathbb{Z}$. □

Remark 1.8. For a t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ on $\mathscr{T}$, note that for $X \in \mathscr{T}$ and $n, m \in \mathbb{Z}$ we have

$$ {}^t\tau^{\leq n}(X[m]) \cong ({}^t\tau^{\leq n+m}X)[m] $$
$$ \text{and } {}^t\tau^{\geq n}(X[m]) \cong ({}^t\tau^{\geq n+m}X)[m]. $$

2. Characterizations of objects for a t-structure

Now we have a nice characterization of the objects in $\mathscr{T}^{\leq n}$:

Corollary 2.1. *Equip $\mathscr{T}$ with a t-structure. For $X \in \mathscr{T}$, the following statements are equivalent:*

(i) $X \in \mathscr{T}^{\leq n}$.

(ii) ${}^t\tau^{\geq n+1}X = 0$.

(iii) *There is a canonical isomorphism* ${}^t\tau^{\le n}X \xrightarrow{\sim} X$.

(iv) $\mathrm{Hom}_{\mathscr{T}}(X, Y) = 0$ *for all* $Y \in \mathscr{T}^{\ge n+1}$.

In particular, the zero object of $\mathscr{T}$ *belongs to* $\mathscr{T}^{\le n}$ *for all* $n \in \mathbb{Z}$.

Proof. There is a canonical morphism of distinguished triangles

$$\begin{array}{ccccccc} {}^t\tau^{\le n}X & \longrightarrow & X & \longrightarrow & {}^t\tau^{\ge n+1}X & \longrightarrow & ({}^t\tau^{\le n}X)[1] \\ \downarrow & & \| & & \downarrow & & \downarrow \\ X & = & X & \longrightarrow & 0 & \longrightarrow & X[1] \end{array}$$

uniquely determined by the second vertical arrow.

(ii)⇔(iii) Immediate by five lemma for distinguished triangles.

(iii)⇒(i) Immediate by strict fullness of $\mathscr{T}^{\le n}$.

(i)⇒(iv) Immediate by (T2).

(iv)⇒(i) Assume $X \notin \mathscr{T}^{\le n}$, then the canonical morphism ${}^t\tau^{\le n}X \to X$ is not an isomorphism. By five lemma for distinguished triangles, we have ${}^t\tau^{\ge n+1}X \not\cong 0$. Now the canonical identification

$$\mathrm{Hom}_{\mathscr{T}}(X, {}^t\tau^{\ge n+1}X) = \mathrm{Hom}_{\mathscr{T}}({}^t\tau^{\ge n+1}X, {}^t\tau^{\ge n+1}X) \neq 0$$

by Theorem 1.7 yields a contradiction.

(i)⇒(ii) The dual statement of the previous implication shows that the zero object belongs to $\mathscr{T}^{\ge n+1}$. Therefore, the result follows from Theorem 1.7. □

Similarly, there is a dual statement for $\mathscr{T}^{\ge n+1}$:

Corollary 2.2. *Equip* $\mathscr{T}$ *with a* t*-structure. For* $X \in \mathscr{T}$*, the following statements are equivalent:*

(i) $X \in \mathscr{T}^{\ge n+1}$.

(ii) ${}^t\tau^{\le n}X = 0$.

(iii) *There is a canonical isomorphism* $X \xrightarrow{\sim} {}^t\tau^{\ge n+1}X$.

(iv) $\mathrm{Hom}_{\mathscr{T}}(Y, X) = 0$ *for all* $Y \in \mathscr{T}^{\le n}$.

In particular, the zero object of $\mathscr{T}$ *belongs to* $\mathscr{T}^{\ge n+1}$ *for all* $n \in \mathbb{Z}$.

Now the following definition makes sense:

Definition 2.3. A t-structure on $\mathscr{T}$ is **nondegenerate** if $\bigcap_{n\in\mathbb{Z}} \mathscr{T}^{\le n}$ and $\bigcap_{n\in\mathbb{Z}} \mathscr{T}^{\ge n}$ both contain only the zero object.

Corollary 2.4. *Equip* $\mathscr{T}$ *with a* t*-structure, then* $\mathscr{T}^{\le n}$ *and* $\mathscr{T}^{\ge n}$ *are stable under extensions.*

Proof. We only verify the extension stability for $\mathscr{T}^{\le n}$ as that of $\mathscr{T}^{\ge n}$ is similar. Let

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

be a distinguished triangle in $\mathscr{T}$ with $A, B \in \mathscr{T}^{\le n}$, then

$$\mathrm{Hom}_{\mathscr{T}}(A, {}^t\tau^{\ge n+1}X) = 0 = \mathrm{Hom}_{\mathscr{T}}(B, {}^t\tau^{\ge n+1}X)$$

by (T2). Applying $\mathrm{Hom}_{\mathscr{T}}(-, {}^t\tau^{\geq n+1}X)$ to the distinguished triangle above, we obtain
$$\mathrm{Hom}_{\mathscr{T}}(X, {}^t\tau^{\geq n+1}X) = 0.$$
By Theorem 1.7, we obtain the canonical identifications
$$\mathrm{Hom}_{\mathscr{T}}({}^t\tau^{\geq n+1}X, {}^t\tau^{\geq n+1}X) = \mathrm{Hom}_{\mathscr{T}}(X, {}^t\tau^{\geq n+1}X) = 0.$$
Particularly, this implies $\mathrm{id}_{{}^t\tau^{\geq n+1}X} = 0$, then ${}^t\tau^{\geq n+1}X = 0$. So $X \in \mathscr{T}^{\leq n}$ by Corollary 2.1. $\square$

3. Compatibility of truncation functors

The two adjoint functors derived in Theorem 1.7 enjoy very nice compatibility properties:

Corollary 3.1. *Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$, and let (a, b) be a pair of integers.*

(i) *If $a \leq b$, then there are natural isomorphisms*
$$ {}^t\tau^{\geq a} \circ {}^t\tau^{\geq b} \xrightarrow{\sim} {}^t\tau^{\geq b} \circ {}^t\tau^{\geq a} \xrightarrow{\sim} {}^t\tau^{\geq b}$$
$$\text{and } {}^t\tau^{\leq a} \circ {}^t\tau^{\leq b} \xrightarrow{\sim} {}^t\tau^{\leq b} \circ {}^t\tau^{\leq a} \xrightarrow{\sim} {}^t\tau^{\leq a}.$$

(ii) *If $a > b$, then there are natural isomorphisms*
$$ {}^t\tau^{\geq a} \circ {}^t\tau^{\leq b} \xrightarrow{\sim} {}^t\tau^{\leq b} \circ {}^t\tau^{\geq a} \xrightarrow{\sim} 0.$$

(iii) *There is a natural isomorphism*
$$ {}^t\tau^{\geq a} \circ {}^t\tau^{\leq b} \xrightarrow{\sim} {}^t\tau^{\leq b} \circ {}^t\tau^{\geq a}.$$

Proof. Let $X \in \mathscr{T}$ be arbitrary.

(i) Since ${}^t\tau^{\geq b}X \in \mathscr{T}^{\geq b} \subset \mathscr{T}^{\geq a}$, the canonical morphism ${}^t\tau^{\geq a}({}^t\tau^{\geq b}X) \to {}^t\tau^{\geq b}X$ is an isomorphism by Corollary 2.2. Let $Y \in \mathscr{T}^{\geq b}$, then we have canonical identifications
$$\mathrm{Hom}_{\mathscr{T}}({}^t\tau^{\geq b}({}^t\tau^{\geq a}X), Y) = \mathrm{Hom}_{\mathscr{T}}({}^t\tau^{\geq a}X, Y) = \mathrm{Hom}_{\mathscr{T}}(X, Y) = \mathrm{Hom}_{\mathscr{T}}({}^t\tau^{\geq b}X, Y).$$
It gives a canonical isomorphism ${}^t\tau^{\geq b}({}^t\tau^{\geq a}X) \xrightarrow{\sim} {}^t\tau^{\geq b}X$. We obtain ${}^t\tau^{\geq b} \circ {}^t\tau^{\geq a} \xrightarrow{\sim} {}^t\tau^{\geq a} \circ {}^t\tau^{\geq b} \xrightarrow{\sim} {}^t\tau^{\geq b}$ where the naturality is clear. The dual argument gives the other natural isomorphism.

(ii) Immediate by Corollary 2.1 and 2.2.

(iii) We assume $a \leq b$ by (ii). Consider distinguished triangles
$${}^t\tau^{\leq a-1}X \cong {}^t\tau^{\leq b}({}^t\tau^{\leq a-1}X) \cong {}^t\tau^{\leq a-1}({}^t\tau^{\leq b}X) \longrightarrow {}^t\tau^{\leq b}X \longrightarrow {}^t\tau^{\geq a}({}^t\tau^{\leq b}X) \longrightarrow ({}^t\tau^{\leq a-1}X)[1],$$
$${}^t\tau^{\leq b}X \longrightarrow X \longrightarrow {}^t\tau^{\geq b+1}X \longrightarrow ({}^t\tau^{\leq b}X)[1],$$
$$\text{and } {}^t\tau^{\leq a-1}X \longrightarrow X \longrightarrow {}^t\tau^{\geq a}X \longrightarrow ({}^t\tau^{\leq a-1}X)[1].$$
The octahedral axiom yields a distinguished triangle
$${}^t\tau^{\geq a}({}^t\tau^{\leq b}X) \longrightarrow {}^t\tau^{\geq a}X \longrightarrow {}^t\tau^{\geq b+1}X \longrightarrow ({}^t\tau^{\geq a}({}^t\tau^{\leq b}X))[1].$$
It admits a comparison to the canonical distinguished triangle
$${}^t\tau^{\leq b}({}^t\tau^{\geq a}X) \longrightarrow {}^t\tau^{\geq a}X \longrightarrow {}^t\tau^{\geq b+1}X \longrightarrow ({}^t\tau^{\leq b}({}^t\tau^{\geq a}X))[1].$$
Since ${}^t\tau^{\geq a}({}^t\tau^{\leq b}X) \cong {}^t\tau^{\leq b}({}^t\tau^{\geq a}X) \in \mathscr{T}^{\leq b}$, we have ${}^t\tau^{\geq a}({}^t\tau^{\leq b}X) \in \mathscr{T}^{\leq b}$ by strict fullness of $\mathscr{T}^{\leq b}$. Thus, Theorem 1.7 implies ${}^t\tau^{\geq a}({}^t\tau^{\leq b}X) \xrightarrow{\sim} {}^t\tau^{\leq b}({}^t\tau^{\geq a}X)$ which is clearly natural in X. $\square$

4. Hearts and t-cohomology

Definition 4.1. Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$, and let $\mathscr{C} = \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$ as a full subcategory of $\mathscr{T}$. The category $\mathscr{C}$ is called the **heart** (or **core**) of the t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$.

Remark 4.2. The heart $\mathscr{C}$ is a (nonempty) strictly full subcategory of $\mathscr{T}$ stable under extension.

Definition 4.3. Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$ with heart $\mathscr{C}$. The zeroth **t-cohomology** functor, also called the zeroth **cohomology with respect to the t-structure**, is the functor

$$^tH^0 := {}^t\tau^{\leq 0} \circ {}^t\tau^{\geq 0} : \mathscr{T} \to \mathscr{C}.$$

More generally, for every integer n, the nth t-cohomology functor $^tH^n : \mathscr{T} \to \mathscr{C}$ is given by

$$^tH^n := {}^tH^0 \circ [n] : \mathscr{T} \longrightarrow \mathscr{C}.$$

Remark 4.4. Note that $^tH^0 \cong {}^t\tau^{\geq 0} \circ {}^t\tau^{\leq 0}$ by Corollary 3.1 (iii). This implies that, for every integer n, the target of $^tH^n$ is $\mathscr{C}$ by strict fullness of $\mathscr{T}^{\leq 0}$ and $\mathscr{T}^{\geq 0}$.

Proposition 4.5. *Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a t-structure on $\mathscr{T}$, and let $X \in \mathscr{T}$.*

- (i) *If $X \in \mathscr{T}^{\leq a}$ for some $a \in \mathbb{Z}$, then $X \in \mathscr{T}^{\leq 0}$ if and only if $^tH^i(X) = 0$ for all $i > 0$.*
- (ii) *If $X \in \mathscr{T}^{\geq b}$ for some $b \in \mathbb{Z}$, then $X \in \mathscr{T}^{\geq 0}$ if and only if $^tH^i(X) = 0$ for all $i < 0$.*
- (iii) *If $X \in \mathscr{T}^{\leq a} \cap \mathscr{T}^{\geq b}$ for some $a, b \in \mathbb{Z}$, then $X \in \mathscr{C}$ if and only if $^tH^i(X) = 0$ for all $i \neq 0$.*

Proof. We only prove (ii) because (i) is dual to (ii) and (iii) follows from (i) and (ii).

Without loss of generality, we assume $a < 0$. Take $a = b$ in the distinguished triangle obtained by the octahedral axiom in Corollary 3.1 (iii), i.e.

$$^tH^a(X)[-a] \longrightarrow {}^t\tau^{\geq a}X \longrightarrow {}^t\tau^{\geq a+1}X \longrightarrow {}^tH^a(X)[1-a].$$

Suppose $^tH^i(X) = 0$ for all $i < 0$. In particular, we have $^tH^a(X) = 0$. Since $X \in \mathscr{T}^{\geq a}$, it follows that $X \cong {}^t\tau^{\geq a}X \cong {}^t\tau^{\geq a+1}X \in \mathscr{T}^{\geq a+1}$, so $X \in \mathscr{T}^{\geq a+1}$ by strict fullness of $\mathscr{T}^{\geq a+1}$. Inductively, we obtain $X \in \mathscr{T}^{\geq 0}$. The other direction is obvious. $\square$

Theorem 4.6. *The heart $\mathscr{C}$ of a t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ on $\mathscr{T}$ is an abelian category.*

Proof. We break the proof into several parts. Let $f : X \to Y$ be an arbitrary morphism in $\mathscr{C}$.

The heart $\mathscr{C}$ is an additive category: Given two triangles $X_i \to Y_i \to Z_i \to X_i[1]$ for $i = 1, 2$ in an arbitrary pretriangulated category, they are distinguished if and only if

$$X_1 \oplus X_2 \longrightarrow Y_1 \oplus Y_2 \longrightarrow Z_1 \oplus Z_2 \longrightarrow (X_1 \oplus X_2)[1]$$

is distinguished. Given $X, Y \in \mathscr{C}$, we apply this criterion to the distinguished triangles $X \xrightarrow{=} X \to 0 \to X[1]$ and $0 \to Y \xrightarrow{=} Y \to 0[1]$, and we obtain a distinguished triangle

$$X \to X \oplus Y \to Y \to X[1].$$

By Remark 4.2, we have $X \oplus Y \in \mathscr{C}$.

Construction of cokernels: The morphism $f : X \to Y$ extends to a distinguished triangle

$$Z[-1] \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

By Corollary 2.4, the shifted distinguished triangle $Y \to Z \to X[1] \to Y[1]$ implies

$$Z \in \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq -1}$$

since $Y \in \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0} \subset \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq -1}$ and $X[1] \in \mathscr{T}^{\leq -1} \cap \mathscr{T}^{\geq -1} \subset \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq -1}$. We claim

$$\mathscr{C} \ni {}^tH^0(Z) \cong {}^t\tau^{\geq 0}Z \cong \mathrm{Coker}(f).$$

Denote the composition $Y \to Z \to {}^t\tau^{\geq 0}Z$ by γ. Let $W \in \mathscr{C}$, and we have an exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(X[1], W) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(Z, W) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(Y, W) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathscr{C}}(X, W).$$

Note that $\mathrm{Hom}_{\mathscr{C}}(X[1], W) = 0$ as $X[1] \in \mathscr{T}^{\leq -1}$ and $W \in \mathscr{T}^{\geq 0}$. Let $g \in \mathrm{Hom}_{\mathscr{C}}(Y, W)$ be arbitrary with $g \circ f = 0$. Since ${}^t\tau^{\geq 0}$ is a left adjoint of $\mathscr{T}^{\geq 0} \hookrightarrow \mathscr{T}$, the above exact sequence is then

$$\begin{array}{c} 0 \longrightarrow \mathrm{Hom}_{\mathscr{C}}({}^t\tau^{\geq 0}Z, W) \xrightarrow{-\circ\gamma} \mathrm{Hom}_{\mathscr{C}}(Y, W) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathscr{C}}(X, W) \\ \qquad\qquad\qquad\qquad\qquad g \longmapsto 0 = g \circ f. \end{array}$$

By exactness, there exists a unique morphism $\overline{g} \in \mathrm{Hom}_{\mathscr{C}}({}^t\tau^{\geq 0}Z, W)$ such that $\overline{g} \circ \gamma = g$.

Construction of kernels: We claim

$$\mathscr{C} \ni {}^tH^0(Z[-1]) \cong {}^t\tau^{\leq 0}(Z[-1]) \cong \mathrm{Ker}(f).$$

Take ${}^t\tau^{\leq 0}(Z[-1]) \to X$ to be the composition ${}^t\tau^{\leq 0}(Z[-1]) \to Z[-1] \to X$. Let $e \in \mathrm{Hom}_{\mathscr{C}}(W, X)$ be arbitrary with $e \circ f = 0$. Consider the exact sequence

$$\begin{array}{c} \mathrm{Hom}_{\mathscr{C}}(W, Y[-1]) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(W, Z[-1]) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(W, X) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathscr{C}}(W, Y) \\ \qquad\qquad\qquad\qquad\qquad e \longmapsto f \circ e = 0. \end{array}$$

Note that $\mathrm{Hom}_{\mathscr{C}}(W, Y[-1]) = 0$, and we identify $\mathrm{Hom}_{\mathscr{C}}(W, {}^t\tau^{\leq 0}(Z[-1])) = \mathrm{Hom}_{\mathscr{C}}(W, Z[-1])$. Thus, the claim is established similarly.

There is a canonical isomorphism $\mathrm{Coim}(f) = \mathrm{Im}(f)$: Recall

$$\mathrm{Coim}(f) = \mathrm{Coker}(\mathrm{Ker}(f) \to X) \text{ and } \mathrm{Im}(f) = \mathrm{Ker}(Y \to \mathrm{Coker}(f)).$$

Extend the composition $Y \to Z \to {}^t\tau^{\geq 0}Z$ to a distinguished triangle $Y \to {}^t\tau^{\geq 0}Z \to I[1] \to Y[1]$, and define I up to isomorphism. We first show $I \in \mathscr{C}$. As ${}^t\tau^{\geq 0}Z \in \mathscr{T}^{\geq 0} \subset \mathscr{T}^{\geq -1}$ and $Y[1] \in \mathscr{T}^{\geq -1}$, we conclude from the shifted distinguished triangle ${}^t\tau^{\geq 0}Z \to I[1] \to Y[1] \to ({}^t\tau^{\geq 0}Z)[1]$ that $I[1] \in \mathscr{T}^{\geq -1}$. So we have $I \in \mathscr{T}^{\geq 0}$. The distinguished triangles

$$Y \longrightarrow Z \longrightarrow X[1] \longrightarrow Y[1]$$

$$\text{and } {}^t\tau^{\leq -1}Z \longrightarrow Z \longrightarrow {}^t\tau^{\geq 0}Z \longrightarrow ({}^t\tau^{\leq -1}Z)[1]$$

together with the previous one imply by the octahedral axiom a distinguished triangle

$$X \longrightarrow I \longrightarrow {}^t\tau^{\leq -1}Z \longrightarrow X[1]$$

with $X, ({}^t\tau^{\leq -1}Z)[-1] \in \mathscr{T}^{\leq 0}$, i.e.

$$\mathrm{Ker}(f) \cong {}^t\tau^{\leq 0}(Z[-1]) = ({}^t\tau^{\leq -1}Z)[-1] \longrightarrow X \longrightarrow I \longrightarrow \mathrm{Ker}(f)[1].$$

It follows that $I \in \mathscr{T}^{\leq 0}$ and so $I \in \mathscr{C}$. The above distinguished triangle yields

$$\mathrm{Coim}(f) = \mathrm{Coker}(\mathrm{Ker}(f) \to X) \cong {}^t\tau^{\geq 0}I \cong I.$$

On the other hand, the distinguished triangle

$$Y \longrightarrow \operatorname{Coker}(f) \cong {}^t\tau^{\geq 0} Z \longrightarrow I[1] \longrightarrow Y[1]$$

implies

$$\operatorname{Im}(f) = \operatorname{Ker}(Y \to \operatorname{Coker}(f)) \cong {}^t\tau^{\leq 0}(I[1][-1]) = {}^t\tau^{\leq 0} I \cong I.$$

Therefore, we obtain canonical isomorphisms $\operatorname{Coim}(f) \cong I \cong \operatorname{Im}(f)$. □

Corollary 4.7. *Let $\mathscr{C}$ be the heart of a t-structure on $\mathscr{T}$, and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms in $\mathscr{C}$. The following two conditions are equivalent:*

(i) *We have a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$.*

(ii) *There exists a morphism $h : Z \to X[1]$ in $\mathscr{T}$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle.*

Moreover, if these conditions hold, then h is unique.

Proof. The uniqueness follows from $\operatorname{Hom}_{\mathscr{C}}(X, Z[-1]) = 0$ since $X \in \mathscr{T}^{\leq 0}$ and $Z[-1] \in \mathscr{T}^{\geq 1}$.

First let the sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be short exact. Extend the morphism f to a distinguished triangle

$$W[-1] \longrightarrow X \xrightarrow{f} Y \longrightarrow W \longrightarrow X[1].$$

Since $Y \in \mathscr{C}$ and $X[1] \in \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq -1}$, we have $W \in \mathscr{T}^{\leq 0}$ and $W[-1] \in \mathscr{T}^{\geq 0}$ by Corollary 2.4. The proof of Theorem 4.6 shows

$$0 = \operatorname{Ker}(f) \cong {}^t\tau^{\leq 0}(W[-1])$$
$$\text{and } (Y \xrightarrow{g} Z) \cong \operatorname{Coker} f \cong (Y \longrightarrow W \longrightarrow {}^t\tau^{\geq 0} W).$$

By Corollary 2.2, the former implies $W[-1] \in \mathscr{T}^{\geq 1}$ and so $W \in \mathscr{T}^{\geq 0}$. Now we have $W \in \mathscr{C}$ and

$$(Y \xrightarrow{g} Z) \cong \operatorname{Coker} f \cong (Y \longrightarrow W).$$

So the result is immediate.

Conversely, we have

$$\operatorname{Ker}(f) \cong {}^t\tau^{\leq 0}(Z[-1]) = 0$$
$$\text{and } \operatorname{Coker}(g) \cong {}^t\tau^{\geq 0}(X[1]) = 0$$

since $Z[-1] \in \mathscr{T}^{\geq 1}$ and $X[1] \in \mathscr{T}^{\leq -1}$. On the other hand, we have

$$\operatorname{Ker}(g) \cong {}^t\tau^{\leq 0} X \cong X$$
$$\text{and } \operatorname{Im}(f) \cong \operatorname{Ker}(Y \to {}^t\tau^{\geq 0} Z) \cong \operatorname{Ker}(Y \to Z) \cong {}^t\tau^{\leq 0} X \cong X.$$

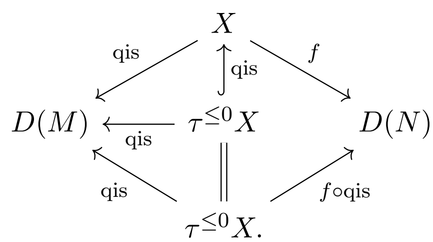
Therefore, the sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact. □

Theorem 4.8. *For an abelian category $\mathscr{A}$, let $\mathscr{T} = D(\mathscr{A})$ be equipped with the natural t-structure. Its heart $\mathscr{C}$ is equivalent to $\mathscr{A}$. It implies $D(\mathscr{C}) \cong \mathscr{T}$, recovering the (original) triangulated category. It also shows that every abelian category can be regarded as a full subcategory of its derived category.*

Proof. Denote the canonical functor $\mathscr{A} \to D(\mathscr{A})$ by D. Let $M, N \in \mathscr{A}$ and $F \in \mathrm{Hom}_{\mathscr{A}}(M, N)$ be arbitrary, then $H^0(D(F)) = F$. So the canonical map $\mathrm{Hom}_{\mathscr{A}}(M, N) \to \mathrm{Hom}_{D(\mathscr{A})}(D(M), D(N))$ is injective. Now let $\varphi \in \mathrm{Hom}_{D(\mathscr{A})}(D(M), D(N))$ be arbitrary and represented by the roof

$$D(M) \xleftarrow{\text{qis}} X \xrightarrow{f} D(N).$$

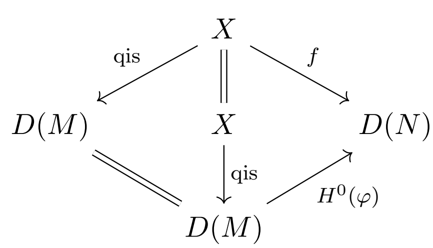
It follows that $H^i(X) = 0$ for all $i \neq 0$. Consider the following commutative diagram



It yields another commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle F^0} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & N & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

for a representative F in $C(\mathscr{A})$ of the homotopy class f in $K(\mathscr{A})$. Since all homotopies from X to $D(N)$ are zero, this representative is unique. In addition, F^0 vanishes on $\mathrm{Im}(d^{-1})$, so it factors through $H^0(F) : H^0(X) \to N$, and we have $H^0(F) = H^0(f) = H^0(\varphi) \circ H^0(\text{qis})$. The commutative diagram



implies $\varphi = D(H^0(\varphi))$. Therefore, the canonical map $\mathrm{Hom}_{\mathscr{A}}(M, N) \to \mathrm{Hom}_{D(\mathscr{A})}(D(M), D(N))$ is surjective. Now the essential image of D is clearly seen to be $\mathscr{C}$. □

However, the nice phenomenon described in Theorem 4.8 is almost never true.

Remark 4.9. Let $\mathscr{C}$ be the heart of a t-structure on $\mathscr{T}$. In general, we have $D(\mathscr{C}) \not\cong \mathscr{T}$.

To construct counterexamples, we consider an abelian category $\mathscr{A}$ and a full abelian subcategory $\mathscr{B}$ both with enough injectives such that $\mathscr{B} \hookrightarrow \mathscr{A}$ is exact. The heart of the natural t-structure on $D_{\mathscr{B}}(\mathscr{A})$ is precisely $\mathscr{B}$. It follows that $D(\mathscr{B}) \not\cong D_{\mathscr{B}}(\mathscr{A})$ frequently happens. The essence of this weird phenomenon breaks down to the investigation of Ext groups, i.e. the Ext group usually differs in a category and a subcategory. More precisely, for $X, Y \in \mathscr{B}$ and $n \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathscr{B})}(X,Y) &= \underline{\mathrm{Ext}}^n_{\mathscr{B}}(X, Y[-n]) \cong \mathrm{Ext}^n_{\mathscr{B}}(X, Y[-n]) \\ \text{and } \mathrm{Hom}_{D_{\mathscr{B}}(\mathscr{A})}(X,Y) = \mathrm{Hom}_{D(\mathscr{A})}(X,Y) &= \underline{\mathrm{Ext}}^n_{\mathscr{A}}(X, Y[-n]) \cong \mathrm{Ext}^n_{\mathscr{A}}(X, Y[-n]). \end{aligned}$$

So $D(\mathscr{B}) \not\cong D_{\mathscr{B}}(\mathscr{A})$ happens whenever $\mathrm{Ext}^1_{\mathscr{B}}(X,Y) \not\cong \mathrm{Ext}^1_{\mathscr{A}}(X,Y)$ for some $X, Y \in \mathscr{B}$.

(i) Let $\mathscr{A} = \mathbb{Z}[x]$-Mod and $\mathscr{B} = \mathbb{Z}$-Mod via $x \mapsto 0$, then $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$ but $\mathrm{Ext}^1_{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}$.

(ii) Let X be an $(n-1)$-connected CW-complex for some $n \geq 2$ such that $\pi_n(X) \neq 0$ is a free abelian group of finite rank, e.g. the complex projective line $\mathbb{P}^1_{\mathbb{C}} \approx S^2$ or the Eilenberg-MacLane space $K(\pi_n(X), n)$. In this case, we have

$$G := H^n_{\mathrm{sing}}(X;\mathbb{C}) \cong \mathrm{Hom}(H_n^{\mathrm{sing}}(X), \mathbb{C}) \cong \mathrm{Hom}(\pi_n(X), \mathbb{C}) \neq 0.$$

Since X is simply connected, there is an equivalence of categories

$$\mathrm{Loc}(X) \cong \mathbb{C}\text{-Mod}$$

by monodromy representation. Consider $D^b_{\mathrm{loc}}(X)$ equipped with the natural t-structure, then its heart $\mathscr{C}$ is canonically equivalent to $\mathrm{Loc}(X) \cong \mathbb{C}$-Mod. We have

$$\mathrm{Hom}_{D(\mathscr{C})}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[n]) \cong \mathrm{Ext}^n_{\mathscr{C}}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X) \cong \mathrm{Ext}^n_{\mathbb{C}}(\mathbb{C},\mathbb{C}) = 0.$$

On the other hand, we have

$$\begin{aligned} \mathrm{Hom}_{D^b_{\mathrm{loc}}(X)}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[n]) &= \mathrm{Hom}_{D(\mathrm{Sh}(X,\mathbb{C}))}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[n]) \\ &\cong \mathrm{Ext}^n_{\mathrm{Sh}(X,\mathbb{C})}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X) \\ &\cong H^n(R\mathrm{Hom}_{\mathrm{Sh}(X,\mathbb{C})}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X)) \\ &= H^n(R\Gamma\mathcal{H}om(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X)) \\ &\cong H^n(R\Gamma(\underline{\mathbb{C}}_X)) \\ &= H^n(X, \underline{\mathbb{C}}_X) \\ &\cong H^n_{\mathrm{sing}}(X;\mathbb{C}) \\ &\cong G. \end{aligned}$$

Theorem 4.10. *The functor ${}^tH^0 : \mathscr{T} \to \mathscr{C}$ is a cohomological functor.*

Proof. Let $X \to Y \to Z \to X[1]$ be a distinguished triangulated in $\mathscr{T}$.

We first assume $X, Y, Z \in \mathscr{T}^{\geq 0}$. There is a canonical commutative diagram

$$\begin{array}{ccccccc} Z[-1] & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ {}^t\tau^{\leq 0}(Z[-1]) & \longrightarrow & {}^t\tau^{\leq 0}X & \longrightarrow & {}^t\tau^{\leq 0}Y & \longrightarrow & {}^t\tau^{\leq 0}Z \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}^tH^0(Z[-1]) & \longrightarrow & {}^tH^0(X) & \longrightarrow & {}^tH^0(Y) & \longrightarrow & {}^tH^0(Z) \end{array}$$

by Theorem 1.7. Let $W \in \mathscr{C}$ be arbitrary, then we obtain

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathscr{C}}(W, Z[-1]) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, Y) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, Z) \\
\uparrow & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr \\
\mathrm{Hom}_{\mathscr{C}}(W, {}^t\tau^{\leq 0}(Z[-1])) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, {}^t\tau^{\leq 0}X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, {}^t\tau^{\leq 0}Y) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, {}^t\tau^{\leq 0}Z) \\
\downarrow & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathrm{Hom}_{\mathscr{C}}(W, {}^tH^0(Z[-1])) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, {}^tH^0(X)) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, {}^tH^0(Y)) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, {}^tH^0(Z))
\end{array}$$

again by Theorem 1.7. Since $\mathrm{Hom}_{\mathscr{C}}(W, Z[-1]) = 0$ with $W \in \mathscr{C}$ arbitrary, we obtain a canonical exact sequence

$$0 \longrightarrow {}^tH^0(X) \longrightarrow {}^tH^0(Y) \longrightarrow {}^tH^0(Z).$$

Dually, if $X, Y, Z \in \mathscr{T}^{\leq 0}$, then we have an exact sequence

$${}^tH^0(X) \longrightarrow {}^tH^0(Y) \longrightarrow {}^tH^0(Z) \longrightarrow 0.$$

Now consider $Z \in \mathscr{T}^{\geq 0}$. Let $W \in \mathscr{T}^{\leq -1}$ be arbitrary, then there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{C}}(W, X) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{C}}(W, Y) \\
\uparrow\wr & & \uparrow\wr \\
\mathrm{Hom}_{\mathscr{C}}(W, {}^t\tau^{\leq -1}X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(W, {}^t\tau^{\leq -1}Y)
\end{array}$$

by $\mathrm{Hom}_{\mathscr{C}}(W, Z[-1]) = 0 = \mathrm{Hom}_{\mathscr{C}}(W, Z)$. So there is a canonical isomorphism ${}^t\tau^{\leq -1}X \xrightarrow{\sim} {}^t\tau^{\leq -1}Y$. The octahedral axiom yields a commutative diagram

$$\begin{array}{ccccccc}
{}^t\tau^{\leq -1}X & \longrightarrow & X & \longrightarrow & {}^t\tau^{\geq 0}X & \longrightarrow & ({}^t\tau^{\leq -1}X)[1] \\
\downarrow\wr & & \downarrow & & \downarrow & & \downarrow\wr \\
{}^t\tau^{\leq -1}Y & \longrightarrow & Y & \longrightarrow & {}^t\tau^{\geq 0}Y & \longrightarrow & ({}^t\tau^{\leq -1}Y)[1] \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Z & = & Z & \longrightarrow & X[1] \\
 & & \downarrow & & \downarrow & & \\
 & & X[1] & \longrightarrow & ({}^t\tau^{\geq 0}X)[1] & &
\end{array}$$

with four distinguished triangles. Apply ${}^tH^0(-)$ to the two columns in the middle, and we obtain a commutative diagram

$$\begin{array}{ccccccc}
 & & {}^tH^0(X) & \longrightarrow & {}^tH^0(Y) & \longrightarrow & {}^tH^0(Z) \\
 & & \downarrow\wr & & \downarrow\wr & & \| \\
0 & \longrightarrow & {}^tH^0({}^t\tau^{\geq 0}X) & \longrightarrow & {}^tH^0({}^t\tau^{\geq 0}Y) & \longrightarrow & {}^tH^0(Z)
\end{array}$$

where the second row is exact. The first two vertical arrows in the above diagram are canonical isomorphisms by Theorem 1.7. So $0 \to {}^tH^0(X) \to {}^tH^0(Y) \to {}^tH^0(Z)$ is an exact sequence.

Dually, if $X \in \mathscr{T}^{\leq 0}$, then we have an exact sequence

$$ {}^tH^0(X) \longrightarrow {}^tH^0(Y) \longrightarrow {}^tH^0(Z) \longrightarrow 0. $$

Finally, we consider the general situation. Denote the canonical composition ${}^t\tau^{\leq 0}X \to X \to Y$ by c. The octahedral axiom gives a commutative diagram

$$
\begin{array}{ccccccc}
{}^t\tau^{\leq 0}X & \xrightarrow{i} & X & \longrightarrow & {}^t\tau^{\geq 1}X & \xrightarrow{c} & ({}^t\tau^{\leq 0}X)[1] \\
\| & & \downarrow{\scriptstyle f} & & \downarrow & & \| \\
{}^t\tau^{\leq 0}X & \xrightarrow{c} & Y & \xrightarrow{u} & W & \longrightarrow & ({}^t\tau^{\leq 0}X)[1] \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle v} & & \downarrow \\
 & & Z & = & Z & \longrightarrow & X[1] \\
 & & \downarrow & & \downarrow & & \\
 & & X[1] & \longrightarrow & ({}^t\tau^{\geq 1}X)[1] & & \\
 & & & & \downarrow & & \\
 & & & & W[1]. & &
\end{array}
$$

Since ${}^t\tau^{\leq 0}X \in \mathscr{T}^{\leq 0}$ and $({}^t\tau^{\geq 1}X)[1] \in \mathscr{T}^{\geq 0}$, we have a commutative diagram

$$
\begin{array}{ccccccc}
{}^tH^0({}^t\tau^{\leq 0}X) & \xrightarrow{{}^tH^0(c)} & {}^tH^0(Y) & \xrightarrow{{}^tH^0(u)} & {}^tH^0(W) & \longrightarrow & 0 \\
{\scriptstyle {}^tH^0(i)}\downarrow\wr & \nearrow{\scriptstyle {}^tH^0(f)} & & & & & \\
{}^tH^0(X) & & & & & &
\end{array}
$$

with an exact row and an exact sequence

$$ 0 \longrightarrow {}^tH^0(W) \xrightarrow{{}^tH^0(v)} {}^tH^0(Z) \longrightarrow {}^tH^0(({}^t\tau^{\geq 1}X)[1]). $$

Note that ${}^tH^0(v) : {}^tH^0(W) \to {}^tH^0(Z)$ is a monomorphism, so $\operatorname{Ker}({}^tH^0(g)) = \operatorname{Ker}({}^tH^0(u))$. Since ${}^tH^0(g) = {}^tH^0(v) \circ {}^tH^0(u)$, they glue to an exact sequence

$$ {}^tH^0(X) \longrightarrow {}^tH^0(Y) \longrightarrow {}^tH^0(Z). \qquad \square $$

Now we are able to generalize Proposition 4.5:

Proposition 4.11. *Let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a nondegenerate t-structure on $\mathscr{T}$ with heart $\mathscr{C}$. Let $n \in \mathbb{Z}$.*

(i) *Let $X \in \mathscr{T}$ with ${}^tH^i(X) = 0$ for all $i \in \mathbb{Z}$, then we have $X = 0$.*

(ii) *A morphism $f : X \to Y$ in $\mathscr{T}$ is an isomorphism if and only if ${}^tH^i(f) : {}^tH^i(X) \to {}^tH^i(Y)$ is an isomorphism for all $i \in \mathbb{Z}$.*

(iii) *Let $X \in \mathscr{T}$, then $X \in \mathscr{T}^{\leq n}$ if and only if ${}^tH^i(X) = 0$ for all $i > n$.*

(iv) *Let* $X \in \mathscr{T}$, *then* $X \in \mathscr{T}^{\geq n}$ *if and only if* ${}^tH^i(X) = 0$ *for all* $i < n$.

(v) *Let* $X \in \mathscr{T}$. *We have* $X \in \mathscr{C}[-n] = \mathscr{T}^{\leq n} \cap \mathscr{T}^{\geq n}$ *if and only if* ${}^tH^i(X) = 0$ *for all* $i \neq n$. *In particular, we have* $X \in \mathscr{C}$ *if and only if* ${}^tH^i(X) = 0$ *for all* $i \neq 0$.

Proof. We only prove (i), (ii), and (iii) since (iv) is dual to (iii) and (v) follows from (iii) and (iv).

(i) First suppose $X \in \mathscr{T}^{\geq 0}$, then ${}^t\tau^{\leq 0}X \cong {}^tH^0(X) = 0$ yields $X \in \mathscr{T}^{\geq 1}$. Inductively, we obtain $X \in \mathscr{T}^{\geq i}$ for all $i \geq 0$. Since the t-structure on $\mathscr{T}$ is nondegenerate, the object $X \in \bigcap_{n\in\mathbb{Z}} \mathscr{T}^{\geq n} = \{0\}$ is the zero object. Similarly, if $X \in \mathscr{T}^{\leq 0}$, we also have $X = 0$. Note that ${}^tH^i({}^t\tau^{\leq 0}X) \cong {}^t\tau^{\leq 0}({}^tH^i(X)) = 0$ and ${}^tH^i({}^t\tau^{\geq 1}X) \cong {}^t\tau^{\geq 1}({}^tH^i(X)) = 0$ hold for all $i \in \mathbb{Z}$. The general case follows from the canonical distinguished triangle ${}^t\tau^{\leq 0}X \to X \to {}^t\tau^{\geq 1}X \to ({}^t\tau^{\leq 0}X)[1]$.

(ii) We only assume ${}^tH^i(f)$ is an isomorphism for all $i \in \mathbb{Z}$ because the other direction is clear. Extend $f : X \to Y$ to a distinguished triangle $X \to Y \to Z \to X[1]$. By Theorem 4.10, apply the cohomological functor ${}^tH^0$, and we obtain an exact sequence

$$\cdots \longrightarrow {}^tH^0(X) \xrightarrow{{}^tH^0(f)} {}^tH^0(Y) \longrightarrow {}^tH^0(Z) \longrightarrow {}^tH^0(X[1]) \xrightarrow{{}^tH^1(f)} {}^tH^0(Y[1]) \longrightarrow \cdots$$

where ${}^tH^i(f)$ is an isomorphism for all $i \in \mathbb{Z}$. So ${}^tH^i(Z) = 0$ for all $i \in \mathbb{Z}$. This implies $Z = 0$.

(iii) It suffices to assume $n = 0$ by the shift functor.

If ${}^tH^i(X) = 0$ for all $i > 0$, then ${}^tH^i({}^t\tau^{\geq 1}X) \cong {}^t\tau^{\geq 1}({}^tH^i(X)) = 0$ for all $i \in \mathbb{Z}$. Indeed, we have ${}^tH^i(X) = 0$ for $i > 0$ and ${}^tH^i(X) \in \mathscr{T}^{\leq 0}$ for $i \leq 0$. Then ${}^t\tau^{\geq 1}X = 0$ and so $X \in \mathscr{T}^{\leq 0}$ by (i).

Conversely, if $X \in \mathscr{T}^{\leq 0}$, then ${}^t\tau^{\geq 1}X = 0$. It follows that ${}^tH^i({}^t\tau^{\geq 1}X) = 0$ for all $i \in \mathbb{Z}$. Since ${}^tH^i(X) \in \mathscr{T}^{\geq 1}$ for $i > 0$, we obtain ${}^tH^i(X) \cong {}^t\tau^{\geq 1}({}^tH^i(X)) \cong {}^tH^i({}^t\tau^{\geq 1}X) \cong 0$ for all $i > 0$. □

5. t-exactness

Definition 5.1. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be triangulated categories equipped with t-structures $(\mathscr{T}_1^{\leq 0}, \mathscr{T}_1^{\geq 0})$ and $(\mathscr{T}_2^{\leq 0}, \mathscr{T}_2^{\geq 0})$ respectively. A triangulated functor $F : \mathscr{T}_1 \to \mathscr{T}_2$ is **left t-exact** (resp. **right t-exact**) if $F(\mathscr{T}_1^{\geq 0}) \subset \mathscr{T}_2^{\geq 0}$ (resp. $F(\mathscr{T}_1^{\leq 0}) \subset \mathscr{T}_2^{\leq 0}$). It is **$t$-exact** if it is both left and right t-exact.

Remark 5.2. If $F : \mathscr{T}_1 \to \mathscr{T}_2$ is left t-exact (resp. right t-exact), then we have $F(\mathscr{T}_1^{\geq n}) \subset \mathscr{T}_2^{\geq n}$ (resp. $F(\mathscr{T}_1^{\leq n}) \subset \mathscr{T}_2^{\leq n}$) for all $n \in \mathbb{Z}$.

Proposition 5.3. *Let* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be triangulated categories equipped with t-structures whose hearts are denoted by* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *respectively. Let* $F : \mathscr{T}_1 \to \mathscr{T}_2$ *be a left t-exact (resp. right t-exact) triangulated functor. Denote* $\mathscr{C}_1 \hookrightarrow \mathscr{T}_1$ *by* ε_1, *and set* ${}^tF = {}^tH^0 \circ F \circ \varepsilon_1 : \mathscr{C}_1 \to \mathscr{C}_2$.

(i) *We have a natural transformation* ${}^tH^0 \circ F \cong {}^tF \circ {}^tH^0 : \mathscr{T}_1^{\geq 0} \to \mathscr{C}_2$ *(resp.* $\mathscr{T}_1^{\leq 0} \to \mathscr{C}_2$*).*

(ii) *The functor* ${}^tF : \mathscr{C}_1 \to \mathscr{C}_2$ *is left exact (resp. right exact).*

(iii) *If F is t-exact, then we obtain a natural transformation* ${}^tH^0 \circ F \cong {}^tF \circ {}^tH^0 : \mathscr{C}_1 \to \mathscr{C}_2$ *and an exact functor* ${}^tF : \mathscr{C}_1 \to \mathscr{C}_2$.

Proof. It suffices to show the results for left t-exact functors, and (iii) is immediate by (i) and (ii).

(i) Recall ${}^tH^0 = {}^t\tau^{\leq 0} \circ {}^t\tau^{\geq 0} : \mathscr{T}_i \to \mathscr{C}_i$ for $i = 1, 2$. For $X \in \mathscr{T}_1^{\geq 0}$, the distinguished triangle

$${}^tH^0(X) \longrightarrow X \longrightarrow {}^t\tau^{\geq 1}X \longrightarrow {}^tH^0(X)[1]$$

implies another distinguished triangle

$$F({}^tH^0(X)) \longrightarrow F(X) \longrightarrow F({}^t\tau^{\geq 1}X) \longrightarrow F({}^tH^0(X))[1]$$

with $F({}^t\tau^{\geq 1}X) = F(({}^t\tau^{\geq 1}X)[1])[-1] \in \mathscr{T}_2^{\geq 1}$ by left t-exactness of F. Applying ${}^tH^0$ to the above yields the canonical isomorphism

$${}^tF({}^tH^0(X)) = {}^tH^0(F({}^tH^0(X))) \cong {}^tH^0(F(X))$$

since ${}^tH^0(F({}^t\tau^{\geq 1}X)) = 0$. Now the naturality is clear from the above.

(ii) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $\mathscr{C}_1$. It gives rise to a distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathscr{T}_1$ by Corollary 4.7. Then we have another distinguished triangle

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F(X)[1].$$

Since $F(X), F(Y), F(Z) \in \mathscr{T}_2^{\geq 0}$, there is an exact sequence

$$0 \longrightarrow {}^tH^0(F(X)) \longrightarrow {}^tH^0(F(Y)) \longrightarrow {}^tH^0(F(Z))$$

by Theorem 4.10. Now it remains to apply (i). □

Definition 5.4. Let $F : \mathscr{T}_1 \to \mathscr{T}_2$ be a triangulated functor between triangulated categories equipped with t-structures. If F is left t-exact with $F(\mathscr{T}_1^{\leq 0}) \subset \mathscr{T}_2^{\leq d}$, then F has **cohomological t-dimension** $\leq d$. If F is right t-exact with $F(\mathscr{T}_1^{\geq 0}) \subset \mathscr{T}_2^{\geq -d}$, then F has **cohomological t-dimension** $\leq d$.

Proposition 5.5. *Let $F : \mathscr{T}_1 \to \mathscr{T}_2$ be a triangulated functor between triangulated categories equipped with t-structures.*

(i) *If F is left t-exact with cohomological dimension $\leq d$, then*

$${}^tH^{n+d}(F(X)) \cong {}^tH^d(F({}^tH^n(X))) \text{ for all } X \in \mathscr{T}_1^{\leq n}.$$

(ii) *If F is right t-exact with cohomological dimension $\leq d$, then*

$${}^tH^{n-d}(F(X)) \cong {}^tH^{-d}(F({}^tH^n(X))) \text{ for all } X \in \mathscr{T}_1^{\geq n}.$$

Proof. We only prove (i) as (ii) is dual to (i). First note that

$${}^tH^n(X) = {}^t\tau^{\leq 0}\,{}^t\tau^{\geq 0}(X[n]) = {}^t\tau^{\leq 0}(({}^t\tau^{\geq n}X)[n]) = ({}^t\tau^{\leq n}\,{}^t\tau^{\geq n}X)[n].$$

So for $X \in \mathscr{T}_1^{\leq n}$ we have

$${}^tH^d(F({}^tH^n(X))) = {}^tH^d(F({}^t\tau^{\leq n}\,{}^t\tau^{\geq n}X)[n]) = {}^tH^{n+d}(F({}^t\tau^{\leq n}\,{}^t\tau^{\geq n}X)) \cong {}^tH^{n+d}(F({}^t\tau^{\geq n}X)).$$

It suffices to show $F({}^t\tau^{\geq n}X) \cong F(X)$. Let $W \in \mathscr{T}_2^{\geq n}$ be arbitrary, then the distinguished triangle

$${}^t\tau^{\leq n-1}X \longrightarrow X \longrightarrow {}^t\tau^{\geq n}X \longrightarrow ({}^t\tau^{\leq n-1}X)[1]$$

implies the exact sequence

$$\begin{aligned}\operatorname{Hom}_{\mathscr{T}_2}((F({}^t\tau^{\leq n-1}X))[1], W) &\longrightarrow \operatorname{Hom}_{\mathscr{T}_2}(F({}^t\tau^{\geq n}X), W)\\ \longrightarrow \operatorname{Hom}_{\mathscr{T}_2}(F(X), W) &\longrightarrow \operatorname{Hom}_{\mathscr{T}_2}(F({}^t\tau^{\leq n-1}X), W)\end{aligned}$$

with $(F({}^t\tau^{\leq n-1}X))[1], F({}^t\tau^{\leq n-1}X) \in \mathscr{T}^{\leq n-1}$. Hence, the first and last terms vanish. We obtain a canonical identification $\mathrm{Hom}_{\mathscr{T}_2}(F({}^t\tau^{\geq n}X), W) = \mathrm{Hom}_{\mathscr{T}_2}(F(X), W)$. Therefore, there is a canonical isomorphism $F({}^t\tau^{\geq n}X) \xrightarrow{\sim} F(X)$. □

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