TALK 9: TRIANGULATED CATEGORIES

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1. Introduction

In this talk we shall discuss some basic notions pertaining to triangulated categories. We have already seen everything in this talk except the Octahedral Axiom (Tr_4) in the examples of $D^b(X), D^b_c(X), D_{loc}(X)$ for X a complex variety.

2. Definitions

We first recall additive categories.

Definition 1. A category $\mathcal{C}$ is **additive** if:
- $\mathcal{C}$ is enriched over abelian groups and composition is bilinear;
- $\mathcal{C}$ has a zero object;
- $\mathcal{C}$ has finite coproducts.

Definition 2. A **triangulated category** is the data of an additive category $\mathcal{K}$ and automorphism $[1] : \mathcal{K} \to \mathcal{K}$ called the **shift functor** and a collection of diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

called **distinguished triangles** subject to the following axioms:

(Tr_1) Any morphism $f : X \to Y$ can be completed to a distinguished triangle

$$X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} X[1].$$

For any $X \in \mathcal{K}$ the diagram

$$X \xrightarrow{1_X} X \to 0 \to X[1]$$

is a distinguished triangle. If we have a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle p[1]}\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

where the top row is a distinguished triangle and the vertical maps are isomorphisms, then the bottom row is also a distinguished triangle;

(Tr_2) (Rotation) The diagram $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} X[1]$ is a distinguished triangle if and only if $Y \overset{g}{\to} Z \overset{h}{\to} X[1] \xrightarrow{-f[1]} Y[1]$ is a distinguished triangle;

(Tr_3) (Completion) Given a solid commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\vdots & & {\scriptstyle p[1]}\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

in which the rows are distinguished triangles there exists a dotted arrow $Z \overset{r}{\to} Z'$ making the whole diagram commute;

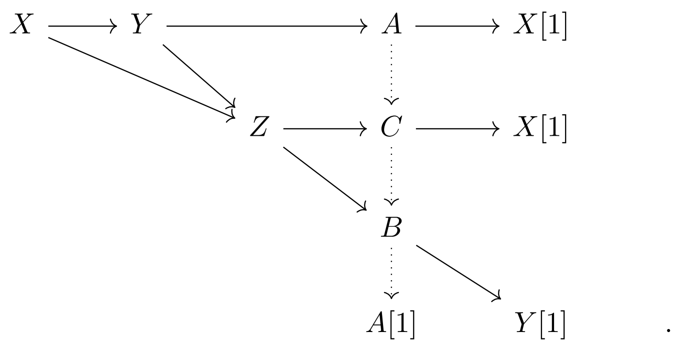
(Tr_4) (Otahedral) Suppose

$$X \to Y \to A \to X[1]$$
$$Y \to Z \to B \to Y[1]$$
$$X \to Z \to C \to X[1]$$

are distinguished triangles then there is a distinguished triangle

$$A \to C \to B \to A[1]$$

making the following diagram commute:



When writing about triangulated categories we shall suppress the shift functor and the distinguished triangles.

Example 1. Let X be a complex variety, then $D^b(X), D^b_c(X), D_{loc}(X)$ are examples where shift is given by shifting cohomological degree. The cone of every morphism gives a triangle where

$$\mathrm{cone}(X \xrightarrow{f} Y)^i = X^{i+1} \oplus Y^i$$

and $d^i_{\mathrm{cone}(X \xrightarrow{f} Y)} = \begin{pmatrix} d_X^{i+1} & 0 \\ f^{i+1} & d_Y \end{pmatrix}$. Furthermore, suppose $U \xrightarrow{j} X$ is open and $Z \xrightarrow{i} X$ is its complement, then for $F \in D^b(X), D^b_c(X), D_{loc}(X)$ we obtain distinguished triangles

$$i_! i^* F \to F \to j_* j^* F \to i_! i^* F[1]$$
$$j_! j^* F \to F \to i * i^* F \to j_! j^* F[1].$$

Definition 3. Suppose $\mathcal{K}, \mathcal{J}$ are triangulated categories an additive functor $F : \mathcal{K} \to \mathcal{J}$ is **triangulated** if:

- For all $X \in \mathcal{K}$ there exists a natural isomorphism $\phi_X : F(X[1]) \to F(X)[1]$;
- For any distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathcal{K}$ the diagram

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(X)[1]$$

is a distinguished triangle in $\mathcal{J}$.

Lemma 1. Suppose

$$\mathcal{K} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{J}$$

is an adjunction and both functors are additive. Then F is triangulated if and only if G is.

Proof. Omitted. □

3. Basic Properties

In this section we shall discuss some basic properties of triangulated categories.

Lemma 2. In a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$, we have $g \circ f = 0$, $h \circ g = 0$, and $-f[1] \circ h = 0$.

Proof. We shall prove that $g \circ f = 0$ and the other cases can be shown analogously after applying rotation. We consider the solid diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{1_X} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ {\scriptstyle 1_X}\downarrow & & {\scriptstyle f}\downarrow & & \vdots & & \downarrow{\scriptstyle 1_{X[1]}} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & X[1] \end{array}$$

then by (Tr_3) the dotted arrow exists making the diagram commute. Hence $g \circ f = 0$. □

Definition 4. Let $\mathcal{K}$ be a triangulated category, and $\mathcal{A}$ an abelian category. An additive functor $F : \mathcal{K} \to \mathcal{A}$ is called a **cohomological functor** if for every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$, the sequence

$$\ldots \longrightarrow F(Z[-1]) \xrightarrow{F(-h[-1])} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

$$\xrightarrow[F(h)]{} F(X[1]) \xrightarrow[F(-f[1])]{} F(Y[1]) \xrightarrow[F(-g[1])]{} F(Z[1]) \xrightarrow[F(-h[1])]{} F(X[2]) \longrightarrow \ldots$$

is a long exact sequence in $\mathcal{A}$.

Lemma 3. Let $\mathcal{K}$ be a $\mathbb{K}$-linear triangulated category. For any object $X \in \mathcal{K}$ the functors $\mathrm{Hom}(X, -) : \mathcal{K} \to \mathbb{K}\text{-mod}$ and $\mathrm{Hom}(-, X) : \mathcal{K}^{\mathrm{op}} \to \mathbb{K}\text{-mod}$ are cohomological.

Proof. Let $A \in \mathcal{K}$, will show $\mathrm{Hom}(A, -)$ is cohomological (the fact for $\mathrm{Hom}(-, A)$ is formally dual). We consider a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. It suffices to show

$$\mathrm{Hom}(A, X) \xrightarrow{f_*} \mathrm{Hom}(A, Y) \xrightarrow{g_*} \mathrm{Hom}(A, Z)$$

is exact (exactness elsewhere can be proven analogously using rotation). We note $g_* \circ f_* = 0$ since $g \circ f = 0$ and $\mathrm{Hom}(A, -)$ is additive. Given $q \in \mathrm{Hom}(A, Y)$ such that $g \circ q = 0$ we obtain the solid diagram

$$\begin{array}{ccccccc} A & \xrightarrow{1_A} & A & \longrightarrow & 0 & \longrightarrow & A[1] \\ {\scriptstyle p}\vdots & & {\scriptstyle q}\downarrow & & \downarrow & & \vdots{\scriptstyle p[1]} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & X[1] \end{array} .$$

We then apply (TR_2), then (Tr_3) and then (Tr_2) in the other direction, hence obtaining the dotted arrow p such that $f_*(p) = f \circ p = q$. As mentioned above exactness elsewhere can be shown by the same argument after rotation. □

Lemma 4. (a) Suppose we have a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

in which the rows are distinguished triangles. If any two of the maps p, q and r are isomorphisms, then the third is an isomorphism as well.

(b) In a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$, f is an isomorphism if and only if $Z \cong 0$.

Proof. (a) By rotation it suffices to show that p and q being isomorphisms implies r is an isomorphism. We consider the commutative diagram:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle p[1]}\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

and suppose p, q are isomorphisms. We note by **Lemma** 3 $\mathrm{Hom}(A, -) =: F(-)$ is a cohomological functor for all $A \in \mathcal{K}$ and hence we get a commutative diagram with exact rows for all $A \in \mathcal{K}$

$$\begin{array}{ccccccccc} F(X) & \longrightarrow & F(Y) & \longrightarrow & F(Z) & \longrightarrow & F(X[1]) & \longrightarrow & F(Y[1]) \\ {\scriptstyle p_*}\downarrow & & {\scriptstyle q_*}\downarrow & & {\scriptstyle r_*}\downarrow & & {\scriptstyle p[1]_*}\downarrow & & {\scriptstyle q[1]_*}\downarrow \\ F(X') & \longrightarrow & F(Y') & \longrightarrow & F(Z') & \longrightarrow & F(X'[1]) & \longrightarrow & F(Y[1]) \end{array}$$

by the classical 5-lemma r^* is an isomorphism for all A. Hence by the Yoneda lemma r is an isomorphism.

(b) We consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{1_X} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ {\scriptstyle 1_X}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle 0}\vdots & & {\scriptstyle 1_{X[1]}}\downarrow \\ X & \xrightarrow[f]{} & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

where the vertical 0 comes from (Tr_3). By part *b*) f is an isomorphism if and only if 0 is an isomorphism.

□

Definition 5. Suppose $\mathcal{K}$ is triangulated and $f : X \to Y$ is a morphism then a **cone** of f is a member of the isomorphism class of objects Z such that $X \xrightarrow{f} Y \to Z \to X[1]$ is a distinguished triangle.

We are now able to reinterpret the octaherdal axiom, given $X \xrightarrow{f} X' \xrightarrow{f'} X''$ then it simply says

$$\mathrm{cone}(f) \to \mathrm{cone}(f' \circ f) \to \mathrm{cone}(f') \to \mathrm{cone}(f)[1]$$

is a distinguished triangle.

Example 2 (Cone is not functorial)**.** We work in $\mathbf{D}(\mathbb{C})$. We consider the solid commutative diagram (where all maps are canonical)

$$\begin{array}{ccccccc} \mathbb{C} & \longrightarrow & 0 & \longrightarrow & \mathbb{C}[1] & \xrightarrow{1_{\mathbb{C}[1]}} & \mathbb{C}[1] \\ \downarrow & & \downarrow & & {\scriptstyle 1_{\mathbb{C}[1]}}\vdots\ \vdots{\scriptstyle 0} & & \downarrow \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}[1] & \longrightarrow & 0 \end{array}$$

and notice both dotted arrows make the whole diagram commute.

Lemma 5. The following properties of a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ are equivalent:

- There is a map $r : Y \to X$ such that $r \circ f = 1_X$,
- There is a map $s : Z \to Y$ such that $g \circ s = 1_Z$,

- $h = 0$,
- There is an isomorphism $u : Y \to X \oplus Z$ such that the following diagram commutes:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ {\scriptstyle 1_X}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle 1_Z}\downarrow & & {\scriptstyle 1_{X[1]}}\downarrow \\ X & \xrightarrow[\mathrm{in}_1]{} & X \oplus Z & \xrightarrow[\mathrm{pr}_2]{} & Z & \xrightarrow[0]{} & X[1] \end{array} .$$

Definition 6. A distinguished triangle satisfying the equivalent conditions of the above lemma is said to be **split**.

Lemma 6. Suppose we have two distinguished triangles and a map q as shown below:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle p[1]}\downarrow \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array} .$$

If $g' \circ q \circ f = 0$, then there exist morphisms p, r making the diagram commute. Moreover if $\mathrm{Hom}(X, Z'[-1]) = 0$, then p is uniquely determined by the condition that $f' \circ p = q \circ f$, and r is uniquely determined by the condition that $r \circ g = g' \circ q$.

Proof. We apply $\mathrm{Hom}(X, -) =: F(-)$ to the bottom row, obtaining an exact sequence:

$$\ldots \longrightarrow F(Z'[-1]) \longrightarrow F(X') \xrightarrow{f'_*} F(Y') \xrightarrow{g'_*} F(Z') \longrightarrow \ldots .$$

By assumption, the element $(X \xrightarrow{q \circ f} Y') \in \ker(g'_*)$ so it is $f'_*(p)$ for some $X \xrightarrow{p} Y$. That is, there is a map $p : X \to X'$ such that $f' \circ p = q \circ f$. The existence of r then follows by (Tr_3).
If $\mathrm{Hom}(X, Z'[-1]) = 0$ then $\mathrm{Hom}(X, X') \hookrightarrow \mathrm{Hom}(X, Y')$ and thus p is uniquely determined by its image in $\mathrm{Hom}(X, Y')$. For uniqueness of r we argue similarly instead applying the functor $F(-) = \mathrm{Hom}(-, Z')$ to the top row. □

Corollary 1. Given two morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, if $\mathrm{Hom}(X, Z[-1]) = 0$, then there is at most one morphism $Z \xrightarrow{h} X[1]$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle.

Proof. Suppose we had two morphisms $h, h' : Z \to X[1]$ that both gave distinguished triangles. Consider the following diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ {\scriptstyle p}\downarrow & & {\scriptstyle 1_Y}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle p[1]}\downarrow \\ X & \xrightarrow[f]{} & Y & \longrightarrow & Z & \xrightarrow[h']{} & X[1] \end{array} .$$

The above lemma tells us there are unique morphisms p and r making this commute, and that they are completely determined by requiring the left and middle squares to commute. Hence $p = 1_X$ and $r = 1_Z$ are forced so commutativity of the right square shows $h = h'$. □