Verdier Duality

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1 Properties of Verdier Duality

Let X, Y denote varieties. Throughout let $f : X \to Y$ be a morphism of varieties.

Theorem 1.1. *There is a contravariant endofunctor* $\mathbb{D} : D^b_c(X) \to D^b_c(X)$ *such that*

(a) If $X = pt$, $\mathbb{D} : D^b_c(pt) \to D^b_c(pt)$ *is linear duality* $M^\bullet \mapsto (M^{-\bullet})^*$

(b) There is a natural isomorphism $\mathbb{D}^2 \xrightarrow{\cong} \mathrm{id}$

(c) For $\mathcal{F} \in D^b_c(X, \mathbb{C}), \mathcal{G} \in D^b_c(Y, \mathbb{C})$ *we have isomorphisms*

$$f_*\mathbb{D}\mathcal{F} \cong \mathbb{D}f_!\mathcal{F} \tag{1}$$

$$f^!\mathbb{D}\mathcal{G} \cong \mathbb{D}(f^*\mathcal{G}) \tag{2}$$

$$f_!(\mathbb{D}\mathcal{F}) \cong \mathbb{D}(f_*\mathcal{F}) \tag{3}$$

$$f^*(\mathbb{D}\mathcal{G}) \cong \mathbb{D}(f^!\mathcal{G}) \tag{4}$$

(d) For $\mathcal{F}, \mathcal{G} \in D^b_c(X, \mathbb{C})$ *we have isomorphisms*

$$R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong \mathbb{D}(\mathcal{F} \overset{\mathrm{L}}{\otimes} \mathbb{D}\mathcal{G}) \cong R\mathcal{H}om(\mathbb{D}\mathcal{G}, \mathbb{D}\mathcal{F})$$

Notation. Let $a_X : X \to \{\star\}$ be the unique map.

Definition 1.2. Let $\omega_X := a^!_X \underline{\mathbb{C}}_{pt}$ the dualizing complex.

Observe: $f^!\omega_Y \cong \omega_X$

Definition 1.3. The verdier duality functor is given by

$$\begin{aligned} \mathbb{D}\colon D^-(X, \mathbb{C})^{op} &\to D^+(X, \mathbb{C}) \\ \mathcal{F} &\mapsto R\mathcal{H}om(\mathcal{F}, \omega_X) \end{aligned}$$

Lemma 1.4 (2.8.1)**.**
- *For any X, ω_X lies in $D^b_c(X, \mathbb{C})$.*
- *The verdier duality functor restricts to a functor*

$$\mathbb{D} : D^b_c(X, \mathbb{C})^{op} \to D^b_c(X, \mathbb{C})$$

Proof. The second point follows from the first, as $R\mathcal{H}om$ preserves constructibility (Talk 7). We do Noetherian Induction. If $X = pt$, then $\omega_X = \mathbb{C}$ so the statement is clear.
Let $j : U \hookrightarrow X$ be the inclusion of a smooth open and we assume that the statement holds for ω_Z, where we denote the complemententary embedding $i : Z := X \setminus U \hookrightarrow X$. Then we have a distinguished triangle

$$i_* \underbrace{i^! \omega_X}_{=\omega_Z} \to \omega_X \to j_* \underbrace{j^* \omega_X}_{\omega_U} \to$$

Here $\omega_U = \mathbb{C}[2n]$ by Prop 1.5 is constructible and ω_Z is constructible by induction hypothesis, and as pushforwards preserve constructibility, the middle term of the distinguished triangle is constructible as well. $\square$

Proposition 1.5 (Poincare Duality, Talk 5)**.** *If X is a smooth variety of dimension n, then we have $\omega_X = \mathbb{C}[2n]$.*

Corollary 1.6. Let X be a smooth variety of dimension n. Then $H^{2n-k}(X, \mathbb{C}) \cong H^k_c(X, \mathbb{C})^*$

Proof. We have

$$R\Gamma_c(X, \mathbb{C})^* = R\mathcal{H}om((a_X)_! \mathbb{C}_X, \mathbb{C}) \cong \mathbb{D}((a_X)_! \mathbb{C}_X) \overset{(c1)}{\cong} (a_X)_* (\underbrace{\mathbb{D}\mathbb{C}_X}_{=\omega_X}) = R\Gamma(X, \mathbb{C})[2n]$$

Now take cohomology:

$$H^k_c(X, \mathbb{C})^* = H^{-k}(R\Gamma_c(X, \mathbb{C})^*) = H^{-k}(R\Gamma(X, \mathbb{C})[2n]) = H^{2n-k}(X, \mathbb{C})$$

$\square$

Proof. of (c) 1) 2).

1. $$f_* \mathbb{D}\mathcal{F} \cong f_* R\mathcal{H}om(\mathcal{F}, \underbrace{\omega_X}_{\cong f^! \omega_Y}) = R\mathcal{H}om(f_! \mathcal{F}, \omega_Y)$$

2. Apply $f^! R\mathcal{H}om(\mathcal{G}, \mathcal{F}) \cong R\mathcal{H}om(f^* \mathcal{G}, f^! \mathcal{F})$ to $\mathcal{F} := \omega_Y$

$\square$

Proof. of (c) 3) 4) Using Lemma 1 and verdier duality is an involution (b) we get

$$f_!\mathbb{D}\mathcal{F} \cong \mathbb{D}\mathbb{D}(f_!(\mathbb{D}\mathcal{F})) \overset{(1)}{\cong} \mathbb{D}(f_*\mathbb{D}(\mathbb{D}\mathcal{F})) \cong \mathbb{D}f_*\mathcal{F}$$

and similarly $f^*\mathbb{D}\mathcal{F} \cong \mathbb{D}f^!\mathcal{F}$ □

Warning: We will later prove (c) 3 , 4 again with using (b) in special cases.

Proof. of [(d)] We first check left isomorphism:

$$\begin{aligned} R\mathcal{H}om(\mathcal{F},\mathcal{G}) &\cong R\mathcal{H}om(\mathcal{F}, R\mathcal{H}om(\mathbb{D}\mathcal{G},\omega_X)) \qquad | \text{ Tensor-hom} \\ &= R\mathcal{H}om(\mathcal{F}\overset{\mathrm{L}}{\otimes}\mathbb{D}\mathcal{G},\omega_X) \\ &= \mathbb{D}(\mathcal{F}\overset{\mathrm{L}}{\otimes}\mathbb{D}\mathcal{G}) \end{aligned}$$

because $\mathbb{D}(\mathcal{F}\overset{\mathrm{L}}{\otimes}\mathbb{D}\mathcal{G}) \cong \mathbb{D}(\mathbb{D}\mathcal{G}\overset{\mathrm{L}}{\otimes}\mathbb{D}\mathbb{D}\mathcal{F})$ by (b), the right isomorphism is a special case of the left one. By duality we only have to check the

□

We can finally show, that all our six functors preserve constructibility.

Corollary 1.7. For any $\mathcal{F} \in D^b_c(Y,\mathbb{C})$ and $f : X \to Y$, $f^!\mathcal{F}$ lies in $D^b_c(X,\mathbb{C})$

Proof. We have

$$f^!\mathcal{F} = f^!(\mathbb{D}\mathbb{D}\mathcal{F}) = \mathbb{D}f_*\mathbb{D}\mathcal{F}$$

and both verdier duality and pushforward preserve constructibility. □

2 Verdier duality dualizes and shifts stalks

We want to see how (a) is generalized.

Lemma 2.1. *Let X be a smooth variety of dimension n and $x \in X$. For any $\mathcal{F} \in D^b_{locf}(X,\mathbb{C})$ there is a natural isomorphism*

$$(\mathbb{D}\mathcal{F})_x \xrightarrow{\sim} \mathbb{D}(\mathcal{F}_x)[2n]$$

Proof. First, for any analatic open subset U containing x (where $i_x : \{x\} \hookrightarrow U$), we have a map

$$R\mathcal{H}om(\mathcal{F}|_U, \mathbb{C}_U[2n]) \to i_{x*}R\mathcal{H}om(i_x^*(\mathcal{F}|_U), i_x^*(\mathbb{C}_U[2n])) \qquad (\star)$$

<u>Observe:</u> $(\star)$ is an isomorphism whenever $i_x : \{x\} \to U$ is a homotopy equivalence (apply Yoneda , use that $i_x^* : D^+_{loc}(U) \to D^+_{loc}(\{x\})$ is an equivalence of categories that preserves tensor products) That yields on hyper cohomology a map

$$\mathbf{H}^k(R\mathcal{H}om(\mathcal{F}|_U, \mathbb{C}_U[2n])) \to \mathbf{H}^k(R\mathcal{H}om(\mathcal{F}_x, \mathbb{C}_x[2n])) = H^k(\mathbb{D}(\mathcal{F}_x)[2n])$$

Now, using a statement about cohomology of the stalk from talk 2 and letting U vary gives

$$\begin{aligned} H^k((\mathbb{D}\mathcal{F})_x) &\cong \varinjlim_{U \ni x} \mathbf{H}^k(R\mathcal{H}om(\mathcal{F}, \omega_x)|_U) \\ &\to H^k(\mathbb{D}(\mathcal{F}_x)[2n]) \end{aligned}$$

As X has a basis of contractible neighborhoods, the latter map is an isomorphism. ☐

In the same spirit:

Remark 1. If X is smooth of dimension n, $\mathcal{L} \in \mathrm{Loc}^{ft} X, \mathbb{C}$ is locally free, then

$$\mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee[2n]$$

Proof. Poincare Duality (Prop 1.5) says : $\omega_X \cong \mathbb{C}_X[2n]$. As $\mathcal{L}$ is locally free, we have

$$\mathbb{D}\mathcal{L} = R\mathcal{H}om(\mathcal{L}, \omega) \cong R\mathcal{H}om(\mathcal{L}, \mathbb{C}_X)[2n] \cong \mathcal{H}om(\mathcal{L}, \mathbb{C}_X)[2n] = \mathcal{L}^\vee[2n]$$

☐

3 Proof of verdier duality

Lemma 3.1.

- *If $f : U \hookrightarrow Y$ is the inclusion of an open subset then (c) (4) holds and refines to*

$$(\mathbb{D}\mathcal{G})|_U \cong \mathbb{D}(\mathcal{G}|_U)$$

- *If $f : Z \to Y$ is proper (e.g. inclusion of a closed subset), then (c) (3) holds:*

$$f_!(\mathbb{D}\mathcal{F}) \cong \mathbb{D}(f_*\mathcal{F})$$

Proof. Use that $f^! = f^*$ ($f_!$ is the left adjoint of f^* for f an open immersion), and $f_! = f_*$ for f proper. ☐

Proposition 3.2. *Let $j : U \hookrightarrow X$ be an inclusion of an open smooth irreducible subset. Then for any $\mathcal{F} \in D^b_{locf}(U, \mathbb{C})$ all of (c) holds, i.e. we have $\mathbb{D}(j_*\mathcal{F}) \cong j_!(\mathbb{D}\mathcal{F})$*

Proof. Step 1:
The case where X is smooth and $Z := X \setminus U$ is a divisor with simple normal crossings.

Lemma 3.3. *Let the irreducible components of Z be $Z_1, \dots, Z_k$. For any $x \in U$, setting $\ell := |\{1 \leq i \leq k \mid x \in Z_i\}|.$, there exists an arbitrary small analytic open $V \ni x$, called an normal crossing coordinate chart, with a commutative diagram*

$$\begin{tikzcd} V \cap U \arrow[r, "\phi|_{V\cap U}, \sim"] \arrow[d, hook, "j'"] & (\mathbb{C}^\times)^\ell \times \mathbb{C}^{n-\ell} \arrow[d, hook, "h"] \\ V \arrow[r, "\sim"'] & \mathbb{C}^n = \mathbb{C}^\ell \times \mathbb{C}^{n-\ell} \end{tikzcd}$$

of horizontal biholomorphisms.

A calculation shows the following:

Lemma 3.4. *Moreover for any $\mathcal{F} \in D^b_{locf}((\mathbb{C}^\times)^\ell \times \mathbb{C}^{n-\ell})$, we have $\Gamma_c(h_*\mathcal{F}) = 0$.*

By Lemma 3.1 we already have $(\mathbb{D}(j_*\mathcal{F}))|_U \cong \mathbb{D}\mathcal{F} \cong (j_!(\mathbb{D}\mathcal{F}))))|_U$. So it remains to show $(\mathbb{D}(j_*\mathcal{F}))|_Z = 0$, i.e. $(\mathbb{D}(j_*\mathcal{F}))_x = 0\ \forall x \in Z$.

$$\begin{aligned} H^k((\mathbb{D}(j_*\mathcal{F})_x)) &= \varinjlim_{V \ni x} H^k(R\Gamma(\mathbb{D}(j_*\mathcal{F})|_V)) \\ &\overset{(c,1)}{=} \varinjlim_{V \ni x} H^k(\mathbb{D}(R\Gamma_c(j_*\mathcal{F})|_V)) = 0 \end{aligned}$$

as x has an arbitrary small normal crossing coordinate chart V, we may

$$R\Gamma_c((j_*\mathcal{F})|_V) \cong R\Gamma_c(j'_*(\mathcal{F}|_{V\cap U})) \overset{3.4}{=} 0$$

where we used open base change.

Step 2: The case where X is irreducible.

Theorem 3.5 (Resolution of Singularities)**.** *Let X be a irreducible variety and $U \subset X$ be a nonempty open smooth subset. There exists a proper map $f : \tilde{X} \to X$ such that*

1. *The variety $\tilde{X}$ is smooth*
2. *The map f restricts to an isomorphism $f^{-1}(U) \to U$*
3. *The preimage $f^{-1}(X \setminus U)$ is a divisor with simple normal crossings.*

Define $\tilde{j}$, such that there is a commutative diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, hook] \arrow[d, "\cong"'] & \tilde{X} \arrow[d, "f"] \\ U \arrow[r, hook, "j"'] \arrow[ur, "\tilde{j}"] & X \end{tikzcd}$$

Then we can exploit this factorization of j as follows:

$$\begin{aligned}\mathbb{D}(j_*\mathcal{F}) &= \mathbb{D}(p_*\tilde{j}_*\mathcal{F}) \\ &\overset{3.1}{\cong} p_!\mathbb{D}(\tilde{j}_*\mathcal{F}) && \mid \text{step 1} \\ &= p_!\tilde{j}_!\mathbb{D}(\mathcal{F}) \\ &= j_!\mathbb{D}(\mathcal{F})\end{aligned}$$

<u>Step 3</u>: Factor

$$\begin{tikzcd} U \arrow[r, "\bar{j}"] \arrow[dr, "j"'] & \bar{U} \arrow[d, "i"] \\ & X \end{tikzcd}$$

and then apply Step 2 to $\bar{j}$ ($\bar{U}$ is irreducible as U is irreducible) and Lemma 3.1 to i.

□

Definition 3.6 (Evaluation map)**.** Given $\mathcal{F} \in D_c^b(X, \mathbb{C})$ we call the map corresponding to the identity under the isomorphisms

$$\mathrm{Hom}(\mathbb{D}\mathcal{F}, \mathbb{D}\mathcal{F}) \cong \mathrm{Hom}(\mathbb{D}\mathcal{F} \overset{\mathrm{L}}{\otimes} \mathcal{F}, \omega_X) \cong \mathrm{Hom}(\mathcal{F}, R\mathcal{H}om(\mathbb{D}\mathcal{F}, \omega_X) = \mathrm{Hom}(\mathcal{F}, \mathbb{D}\mathbb{D}\mathcal{F})$$

the evaluation map $ev_{\mathcal{F}} : \mathcal{F} \to \mathbb{D}\mathbb{D}\mathcal{F}$.

Proof. of (b).
<u>Step 1</u> X is smooth of dimension n and irreducible and $\mathcal{F} \in D^b_{locf}(X, \mathbb{C})$. We will show, that the evaluation map induces an isomorphism on stalks $\mathcal{F}_x \to \mathbb{D}(\mathbb{D}\mathcal{F})_x$.
As

$$\mathbb{D}\mathcal{F} \overset{1.5}{=} R\mathcal{H}om(\mathcal{F}, \mathbb{C}[2n])$$

lies in $D^b_{locf}(X, \mathbb{C})$, we can apply Lemma 2.1 twice:

$$\begin{aligned}(\mathbb{D}\mathbb{D}\mathcal{F})_x &\cong R\,\mathrm{Hom}((\mathbb{D}\mathcal{F})_x, \mathbb{C}[2n]) \\ &\cong R\,\mathrm{Hom}(R\,\mathrm{Hom}(\mathcal{F}_x, \mathbb{C}), \mathbb{C}) && |\text{Linear duality} \\ &\cong \mathcal{F}_x\end{aligned}$$

Where in the last step, we used that linear duality on finitely gerated complexes is an involution.
<u>Step 2</u> General X
We use Noetherian Induction: If X is a point we are done by the <u>Step 1</u>.
If not, we can choose a smooth irreducible open subset $j : U \hookrightarrow X$ such that $\mathcal{F}|_U \in D^b_{locf}(U, \mathbb{C})$. Let $i : Z := X \setminus U \hookrightarrow X$. By <u>Step 1</u> and Lemma 3.1, restricting the evaluation map to U

$$j^*\mathcal{F} \xrightarrow{\sim} \mathbb{D}\mathbb{D}(j^*\mathcal{F}) \cong j^*\mathbb{D}\mathbb{D}\mathcal{F} \tag{5}$$

yields an isomorphism.
Furtermore, for any $\mathcal{G} \in D^b_c(Z, \mathbb{C})$, $\mathrm{ev}_{\mathcal{G}}$ is an isomorphism by Noetherian induction, hence also $\mathrm{ev}_{i_*\mathcal{G}}$, beeing the composite

$$i_*\mathcal{G} \xrightarrow{\sim} i_*\mathbb{D}(\mathbb{D}(\mathcal{G})) \overset{3.1}{\cong} \mathbb{D}(\mathbb{D}(i_*\mathcal{G})) \tag{6}$$

Consider the following horizontal distinguished triangles

$$\begin{tikzcd}
j_!j^*\mathcal{F} \arrow[r] \arrow[d, "\mathrm{ev}_{j_!j^*\mathcal{F}}"] \arrow[dd, bend right=60, "{5,\sim}"'] & \mathcal{F} \arrow[r] \arrow[d, "\mathrm{ev}_{\mathcal{F}}"] & i_*i^*\mathcal{F} \arrow[r] \arrow[d, "{\mathrm{ev},\sim \text{ by } 6}"] & {} \\
\mathbb{D}\mathbb{D}j_!j^*\mathcal{F} \arrow[r] \arrow[d, dashed, "d"] & \mathbb{D}\mathbb{D}\mathcal{F} \arrow[r] \arrow[d, equal] & \mathbb{D}\mathbb{D}i_*i^*\mathcal{F} \arrow[r] \arrow[d, dashed] & {} \\
j_!j^*\mathbb{D}\mathbb{D}\mathcal{F} \arrow[r] & \mathbb{D}\mathbb{D}\mathcal{F} \arrow[r] & i_*i^*\mathbb{D}\mathbb{D}\mathcal{F} \arrow[r] & {}
\end{tikzcd}$$

Claim : *In the middle sequence indeed the right term vanishes when restricted to U and the left term is zero when restricted to Z.*
If shown the claim: By Talk 2, part 2, the lower distinguished triangle is the universal distinguished triangle $A \to \mathbb{D}\mathbb{D}\mathcal{F} \to C \to$, such that A vanishes when restricted to Z and C is supported on Z. Hence there are canonical comparison isomorphism d and e. One easily checks that d makes the triangle on the left commute. As the left triangle commutes, $\mathrm{ev}_{j_!j^*\mathcal{F}}$ has to be an isomorphism as well. By two out of three property for the upper two distinguished triangles, $\mathrm{ev}_{\mathcal{F}}$ has to be an isomorphism, as desired.
Proof of the claim.
The first assertion follows by Lemma 3.1.
The second may be proven by the following, using $j^*\mathcal{F} \in D^b_{locf}$:

$$i^*\mathbb{D}\mathbb{D}j_!j^*\mathcal{F} \overset{3.2}{=} i^*j_!\mathbb{D}\mathbb{D}j^*\mathcal{F} = 0$$

(Recall: The last equation holds very generally: Setting $\mathcal{G} = \mathbb{D}\mathbb{D}j^*\mathcal{F}$:

$$(i^*j_!\mathcal{G})_z = (j_!\mathcal{G})_z \overset{Talk3}{=} R\Gamma_c(\mathcal{G}|_{\underbrace{j^{-1}(z)}_{\varnothing}}) = 0$$

□ (Claim)

□

4 Compatibilities with external tensor product

Slogan : external tensor products preserve everything.

Proposition 4.1 (2.9.4)**.** $\mathbb{D}\mathcal{F} \boxtimes \mathcal{G} \cong R\mathcal{H}om(p_1^*\mathcal{F}, p_2^!\mathcal{G})$.

Corollary 4.2. We have natural isomorphisms

- $\mathbb{D}\mathcal{F} \boxtimes \mathbb{D}\mathcal{G} = \mathbb{D}(\mathcal{F} \boxtimes \mathcal{G})$
- $$R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \boxtimes R\mathcal{H}om(\mathcal{F}', \mathcal{G}') = R\mathcal{H}om(\mathcal{F} \boxtimes \mathcal{F}', \mathcal{G} \boxtimes \mathcal{G}')$$

Proof. in Prop 4.1, replace $\mathcal{G}$ by $\mathbb{D}\mathcal{G}$, use Thm 1.1 (c,2) and apply tensor-hom

$$\mathbb{D}\mathcal{F} \boxtimes \mathbb{D}\mathcal{G} \cong R\mathcal{H}om(p_1^*\mathcal{F}, \underbrace{p_2^!\mathbb{D}\mathcal{G}}_{=\mathbb{D}p_2^*\mathcal{G})}) = R\mathcal{H}om(p_1^*\mathcal{F} \overset{\mathrm{L}}{\otimes} p_2^*\mathcal{G}, \omega_Y) = \mathbb{D}(\mathcal{F} \boxtimes \mathcal{G})$$

Actually we dont need the involution property for this.
For the second part use Thm 1.1 (d) to express everything just by (external) tensor products and verdier duals ☐

Proposition 4.3. *Let $f : X \to X', g : Y \to Y'$ be maps of varieties.*

1. *If $\mathcal{F} \in D_c^b(X', \mathbb{C}), \mathcal{G} \in D_c^b(Y', \mathbb{C})$ there are natural isomorphisms*

$$f^*\mathcal{F} \boxtimes g^*\mathcal{G} \cong (f \times g)^*(\mathcal{F} \boxtimes \mathcal{G}) \tag{7}$$
$$f^!\mathcal{F} \boxtimes g^!\mathcal{G} \cong (f \times g)^!(\mathcal{F} \boxtimes \mathcal{G}) \tag{8}$$

2. *If $\mathcal{F} \in D_c^b(X, \mathbb{C}), \mathcal{G} \in D_c^b(Y, \mathbb{C})$ there is a natural isomorphism*

$$f_*\mathcal{F} \boxtimes g_*\mathcal{G} \cong (f \times g)_*(\mathcal{F} \boxtimes \mathcal{G}) \tag{9}$$
$$f_!\mathcal{F} \boxtimes g_!\mathcal{G} \cong (f \times g)_!(\mathcal{F} \boxtimes \mathcal{G}) \tag{10}$$

Proof. (7) is straightforward using that pullback is functorial and preserves the tensor product.
(9) is Proposition 2.9.1 in the Book. From this Now (8) and (10) follow by invoking Corollary 4.2. ☐

Corollary 4.4 (Künneth Formula)**.** There is a natural isomorphism

$$R\Gamma(\mathbb{C}_X) \overset{\mathrm{L}}{\otimes} R\Gamma(\mathbb{C}_Y) \cong R\Gamma(\mathbb{C}_{X \times Y})$$

Proof. As pullbacks of the constant sheaf $\mathbb{C}$ are constant, we have $\mathbb{C}_X \boxtimes \mathbb{C}_Y = \mathbb{C}_{X \times Y}$. So

$$R\Gamma(\mathbb{C}_{X \times Y}) = a_{X \times Y *}(\mathbb{C}_{X \times Y}) = (a_X \times a_Y)_*(\mathbb{C}_X \boxtimes \mathbb{C}_Y) = a_{X*}\mathbb{C}_X \boxtimes a_{Y*}\mathbb{C}_Y$$

by (9). As the external tensor product of sheaves on a point is the same as the ordinary tensor product, we win. ☐

Remark 2. The same works for compactly supported cohomology by applying (10) of Prop 4.3.

Corollary 4.5 (of (8))**.** We have $\omega_X \boxtimes \omega_Y \cong \omega_{X \times Y}$