facts from the 2nd talk

Support of a section $s \in \Gamma(U, F)$ is a closed set and the proper pushforward is a subfunctor of usual pushforward consisting of sections $s \in F(f^{-1}(U))$: $supp(s) \to U$ *is proper*.

For a locally closed immersion $f : Y \to X$ the functor $^\circ f_!$ admits a right adjoint $^\circ f^!$ "restrictions with support in Y". In general $f^!$ doesn't come from abelian level.

Fact 0.0.1. *If $f : Y \to X$ is proper then $f_* = f_!$, in particular this holds if $Y \to X$ is a closed embedding. If $U \to X$ is an open embedding then $f^! = f^*$. This is almost by definitions of the functors and proper map.*

Fact 0.0.2. *Take $F \in D^b_c(X)$. For a closed embedding $i : Z \to X$ and a complementary open embedding $j : U \to X$ we have distinguished triangles*

$$(j_!j^!F =)\ j_!j^*F \to F \to i_*i^*F\ (= i_!i^*F) \tag{1}$$

and

$$(i_!i^!F =)\ i_*i^!F \to F \to j_*j^*F\ (= j_*j^!F). \tag{2}$$

recollection from the 6th talk and more

Fact 0.0.3. *Let $f : Y \to X$ be a locally closed immersion. Then there is a stratification of X such that Y is a union of strata. In particular if F is a constructible sheaf on Y then $f_!(F)$ is constructible: this can be shown by using common refinement of stratifications of X and Y. In fact if $u : U \to X$ is a stratum not meeting Y then $u^*f_!(F)$ is a zero sheaf on U. If $v : V \to X$ and $v' : V \to Y$ is a stratum contained in Y then $v^*f_!(F) = v'^*(F)$, for example by proper base change.*

Combining the above fact with closed-open distinguished triangles from fact 0.0.2 one gets a plesant corollary:

Corollary 0.0.4 (constructibility is local)**.** *If $F \in D^b_c(X)$ and fix closed embedding $Z \to X$ with complement U. Then F is constructible if and only if $F|_Z$ and $F|_U$ are constructible. More generally for a cover of X by locally closed subsets, F is constructible if and only if F restricted to every element of the cover is constructible.*

Definition 0.0.5. *Recall that a **differentiable locally trivial fibration** with fiber F is a map $f : M \to N$ between smooth manifolds such that every $n \in N$ has a neighbourhood $U = U_n$ such that we have a diffeomorphism $g : f^{-1}(U) \simeq F \times U$ such that the following diagram commutes*

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\quad g \quad} & F \times U \\ & {\scriptstyle f|_{f^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle \pi_U} & \\ & U & \end{array} \quad .$$

We state a result from Differential Topology:

Theorem 0.0.6 (Ehresmann's fibration theorem)**.** *Let $f : X \to Y$ be a smooth, surjective, proper morphism of smooth varieties. Then f is a differentiable locally trivial fibration.*

Another classical and important theorem from Algebraic Geometry:

Theorem 0.0.7 (Nagata compactification theorem)**.** *Let $f : X \to Y$ be a morphism between varieties. Then there exists a variety $\widetilde{X}$ such that f can be factored as*

$$\begin{array}{ccc} & \widetilde{X} & \\ {\scriptstyle j} \nearrow & & \searrow {\scriptstyle \widetilde{f}} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

where j comes from a Zariski-open immersion and $\widetilde{f}$ is a proper map.

Now we define what a simple normal crossing divisor is.

Definition 0.0.8. *Let X be a smooth variety of dimension n (in particular we assume that X is equidimensional). Let $Z \subset X$ be a closed subvariety of dimension $n-1$, with irreducible components $Z_1, ..., Z_k$. The variety Z is said to be a* ***divisor with simple normal crossings*** *if the following conditions hold:*

1. *Each Z_i is is smooth.*
2. *For any $x \in X$ define $I_x \subset \{ 1, 2, ..., k\}$ consisting of indices i such that x belongs to Z_i. Then we require that there exists an affine neighbourhood U of x and regular functions $\{f_i\}_{i \in I}$ on U such that*
   (a) $V(f_i) = Z_i \cap U$ for all $i \in I$.
   (b) Differentials $\{df_i(x)\}_{i \in I}$ are linearly independent.

Definition 0.0.9 (notation)**.** *Let $Z \subset X$ be a divisor with irreducible components $Z_1, ..., Z_k$. For any $x \in X$ we define $I_x = \{i \in \{1, 2, ..., k\} : x \in Z_i\}$. For any $I \subset \{1, 2, ..., k\}$ denote*

$$Z_I = \bigcap_{i \in I} Z_i$$

and

$$X_I = \{x \in X : I_x = I\}.$$

Definition 0.0.10. *Let us keep notation as in 0.0.9. Then*

$$\{X_I\}_{I\subset\{1,2,...,k\}}$$

*forms a stratification of X called **normal crossing stratification** (associated to a divisor Z with simple normal crossings).*

Definition 0.0.11. 1. *Let $f : X \to Y$ be a map between smooth varieties and $\bigcup_{i=1}^{k} Z_i = Z \subset X$ be a divisor. We say that f is **transverse to** Z if for all $I \subset \{1, 2, ..., k\}$ such that $Z_I \neq \emptyset$ we have that $f|_{Z_I}$ is smooth and surjective.*

2. *Moreover, in the setting as above, we say that f is a **transverse (to Z) locally trivial fibration** if every point $y \in Y$ admits an analytic neighbourhood V and a a diffeomorphism*

$$b : f^{-1}(U) \to f^{-1}(y) \times U$$

such that

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\quad b \quad} & F \times U \\ & {\scriptstyle f|_{f^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle \pi_U} & \\ & U. & \end{array}$$

commute. We require that b restricts to a diffeomorphism

$$b|_{f^{-1}(U)\cap Z_I} : f^{-1}(U) \cap Z_I \to (f^{-1}(y) \cap Z_I) \times U$$

for each subset $I \subset \{1, 2, ..., k\}$, and that makes

$$\begin{array}{ccc} f^{-1}(U) \cap Z_I & \xrightarrow{\quad b \quad} & (f^{-1}(y) \cap Z_I) \times U \\ & {\scriptstyle f|_{f^{-1}(U)\cap Z_I}} \searrow \quad \swarrow {\scriptstyle \pi_U} & \\ & U & \end{array}$$

commute. In particular $f|_{X\setminus Z}$ is a locally trivial fibration.

Remark 0.0.12 ("proper map vs proper morphism")**.** *Note that if we start with a proper morphism of varieties in the language of Algebraic Geometry (over complex numbers) then it gives us a proper map if we pass to analytic topology.*

Let us recall an algebraic analog of Sard's theorem:

Theorem 0.0.13 (generic smoothness)**.** *Let $f : X \to Y$ be a morphism between varieties and assume that X is smooth. Then there exists an open set $U \subset Y$ such that $f|_{f^{-1}(U)}$ is smooth.*

Theorem 0.0.14. *Let X be an irreducible variety with dense, open and smooth subset U. There exists a smooth variety $\widetilde{X}$ and a proper map $\pi : \widetilde{X} \to X$ such that $\pi^{-1}(U) \to U$ is an isomorphism and $\pi^{-1}(X \setminus U)$ is a divisor with simple normal crossings.*

Lemma 0.0.15 (Lemma 2.4.2. in Achar)**.** *Let X be a smooth variety and let $Z \subset X$ be a divisor with simple normal crossing with irreducible components $Z_1, ..., Z_k$. Consider $\{X_I\}_{I \subset \{1,2,...,k\}}$ (see 0.0.9 and 0.0.10 for the definition of X_I). Then for any $I : X_I \to X$ and any local system $\mathcal{L}$ on X_I we have that $i_*\mathcal{L}$ is constructible with respect to the normal crossing stratification.*

proof sketch. $Z_I \subset X$ locally looks like $\mathbb{A}^n \subset \mathbb{A}^m$ and it is enough to consider the last case which is not hard. □

Lemma 0.0.16 (Lemma 2.4.5. in Achar)**.** *Let $f : X \to Y$ be a smooth morphism of smooth varieties, and let $Z \subset X$ be a divisor with simple normal crossings. Assume that f is a transverse locally trivial fibration with respect to Z. If $F \in D^+(X)$ is (weakly) constructible with respect to the normal crossings stratification, then $f_*F \in D^+_{loc}$.*

Claim 0.0.17. *f^* preserves constructibility. Indeed take $f : Y \to X$, take a sheaf F on X constructible with respect to a stratification L. In order to see the flaim take preimages of strata of L , refine it to a stratification L' on Y. The stratification L' whitnesses constructibility of f^*F.*

0.1 f_* and $f_!$ preserve constructibility

Theorem 0.1.1. *Let $f : X \to Y$ be a map between varieties and let F be an element of $D^b_c(X)$. Then $f_*(F)$ and $f_!(F)$ belong to $D^b_c(Y)$.*

Before giving a proof we reduce it to a special case:

Lemma 0.1.2. *It is enough to treat the case of f_*. We can without loss of generality assume that F is just a sheaf (not a complex of sheaves). We can furthermore assume that X is irreducible and f is dominant.*

Proof of the Lemma 0.1.2.

1. Reduction to treating f_* only: by Nagata compactification 0.0.7 it is enough to consider cases of f being proper and open embedding. For f proper we have $f_* = f_!$. Also, for f open embedding we know that $f_!$ preserves constructibility (see 0.0.3). Thus it is enough to prove that constructibility is preserved under f_*.

2. Reduction to F being a sheaf: this is achived in the similar manner as during talk 6 namely by using truncations, boundedness of F and the fact that D^b_c is a triangulated category.

3. X is irreducible: Suppose X is arbitrary, not necessarily irreducible, and suppose we have the theorem for any f with an irreducible domain. We

will show by induction on number irreducible components X (X is the domain of f) that we have the theorem for all f. Indeed if X is not irreducible then we can consider inclusions

$$i_1 : X_1 \hookrightarrow X \hookleftarrow U_1 : j$$

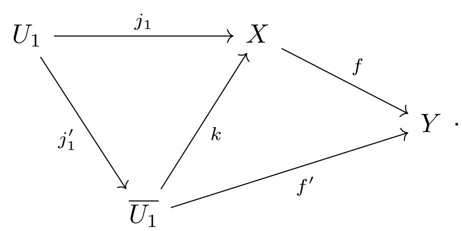
where X_1 is an irreducible component and U_1 is the complement. Then we can form a distinguished triangle

$$j_! j^* F \to F \to i_{1*} i_1^* F$$

(see (1)) and we can apply f_* yielding

$$f_* j_! j^* F \to f_* F \to f_* i_{1*} i_1^* F.$$

Note that $j^* F$ is constructible by Claim 0.0.17 and, as j is in particular locally closed, $j_! j^* F$ is constructible by Lemma 0.0.3. Now we will chase F along the diagram



Note that $f_* j_{1!} j_1^* F = f_*(k_! j_1'{}_!) j_1^* F = (f_* k_*) j_1'{}_! j_1^* F = f'_* j_1'{}_! j_1^*$, we simply use functoriality of proper and usual pushforwards and the fact that k is proper being a closed embedding. We see that $j_1'{}_! j_1^* F$ is constructible by 0.0.3 and 0.0.17. Finally, as $\overline{U_1}$ has less irreducible components than X we deduce that $f'_*(j_1'{}_! j_1^* F)$ is constructible.

4. f is dominant: we can factor f through its image: $X \to \overline{f(X)} \to Y$. We know that pushforward along the last map preserves constructibility as it is a closed immersion (again, see 0.0.3). Thus it is enough to treat dominant maps.

□

proof of Theorem 0.1.1

Take a stratification S trivializing F, and take the stratum U of S which is dense in X (the "biggest" stratum). Recall that U is smooth and $\mathcal{L} := F|_U$ is a local system by construction. Take $Z = X \setminus U$. Using (2) *we see that the maps*

$$U \xrightarrow{h} X \xleftarrow{i} Z$$

induce a distinguished triangle

$$i_*i^!F \to F \to h_*\mathcal{L} \tag{3}$$

and after applying f_* we get

$$f_*i_*i^!F \to f_*F \to f_*h_*\mathcal{L} \tag{4}$$

We will study the last triangle in detail. Using 0.0.4 we see that in order to prove that f_*F is constructible it is enough to prove that $f_*i_*i^!F$ and $f_*h_*\mathcal{L}$ are constructible. The proof proceeds as follows:

step (A) *We first prove that $f_*h_*\mathcal{L}$* ***is constructible in case of*** *f* ***being an open embedding****. By 0.0.14 we can find a proper map $p : \widetilde{Y} \to Y$ such that it is isomorphism over $p^{-1}(U)$ and $\widetilde{Z'} := \widetilde{Y} \setminus U$ is a divisor with simple normal crossings. Denote $\widetilde{h} : U \to \widetilde{Y}$ and $Z := p(\widetilde{Z'})$. We will use 0.0.4 again; it is clear that $(f_*h_*\mathcal{L})|_U$ is constructible being local system and we only have to investigate $(f_*h_*\mathcal{L})|_{Z'}$. For this look at the following diagram:*

$$\begin{array}{ccc} \widetilde{Z'} & \xrightarrow{q'} & \widetilde{Y} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Z' & \xrightarrow{q} & Y \end{array} .$$

Using proper basechange we see that $p'_*q'^*(\widetilde{h}_*\mathcal{L}) = q^*p_*(\widetilde{h}_*\mathcal{L})$. Naturality of pushforward gives us $q^*f_*h_*\mathcal{L} = (q^*p_*)(\widetilde{h}_*\mathcal{L})$. And so $p'_*q'^*(\widetilde{h}_*\mathcal{L}) = q^*f_*h_*\mathcal{L})$ (we can use a notation $(f_*h_*\mathcal{L})|_{Z'}$ for the last expression) is constructible by the following:

1. *$\widetilde{h}_*\mathcal{L}$ is constructible by Lemma 0.0.15.*
2. *$q'^*(\widetilde{h}_*\mathcal{L})$ is constructible by 0.0.17*
3. *By induction on the dimension of the domain of f we know that p'_* preserves constructibility and therefore $p'_*q'^*(\widetilde{h}_*\mathcal{L})$ is constructible.*

step (B) *Now we will prove that,* ***for a general map*** *f,* *$f_*h_*\mathcal{L}$* ***being constructible implies*** *f_*F* ***being constructible****. By the step (A) we know that h_*F is constructible. The middle and the last term in the triangle*

$$i_*i^!F \to F \to h_*\mathcal{L}$$

are constructible and therefore $i_*i^!F$ is constructible as well. As i^* preserves constructibility and counit i^*i_* is the identity on $D^b(Z)$ we see that $i^!F$ is itself constructible. Using induction hypothesis we get that $(fi)_*$ preserves constructibility (as $dim(Z) < dim(X)$).

Remark 0.1.3. *note that combining steps (A) and (B) we see that $f_* F$ is constructible whenever f is an open embedding.*

step (C) ***Case of** f **being proper and** $dim(Y) = 0$ (so Y is just a point). In other words we want global sections to be finitely generated. The second statement of theorem 2.6.2 in Acher (Artin's vanishing theorem) claims precisely that.*

step (D) *We proceed with showing that $f_* h_* \mathcal{L}$ is constructible. We assume f to be proper and $dim(Y) > 0$. Using 0.0.14 we find a map $p : \widetilde{X} \to X$ such that $p^{-1}(U) \to U$ is an isomorphism and $\widetilde{Z} := p^{-1}(Z) = \widetilde{X} \setminus U$ is a divisor with simple normal crossings. Let $\widetilde{Z}_1, ..., \widetilde{Z}_k$ be irreducible components of $\widetilde{Z}$.*

For all $I \subset \{1, 2, ..., k\}$ recall notation form 0.0.9. Take I such that

$$\widetilde{f}|_{\widetilde{Z}_I} : \widetilde{Z}_I \to Y \tag{5}$$

is not dominant and define V_I to be the complement of $\pi(\overline{\widetilde{Z}_I})$. If I is such that the map (5) is dominant then we use Generic Smoothness 0.0.13 to find an open, nonempty $V_I \subset Y$ such that

$$\widetilde{f}|_{\widetilde{f}^{-1}(V_I)} : \widetilde{f}^{-1}(V_I) \cap \widetilde{Z}_I \to V_I$$

is smooth. We define

$$V := \bigcap_{I \subset \{1,2,...,k\}} V_I.$$

and

$$\widetilde{V} := \widetilde{f}^{-1}(V).$$

By Ehresmann's fibration theorem 0.0.6 we see that $\widetilde{f}|_{\widetilde{V}}$ is a locally trivial differentiable fibration. By construction it is transverse to Z (see Def. 0.0.11 for a definition of a transverse (to a divisor Z) locally trivial fibration).

We have that

$$(f_* h_* \mathcal{L})|_V = (\widetilde{f}_* \widetilde{h}_* \mathcal{L})|_V = (\widetilde{f}|_{\widetilde{V}})_* [(\widetilde{h}_* \mathcal{L})|_{\widetilde{V}}], \tag{6}$$

the first equality by naturality, the second by proper (or smooth, since the map is smooth when restricted to $\widetilde{V}$ by construction) base change. We use Lemma 0.0.15 again to see that $\widetilde{h}_* \mathcal{L}$ is constructible. By 0.0.17 we get that $(\widetilde{h}_* \mathcal{L})|_{\widetilde{V}}$ is constructible as well. By Lemma 0.0.16 we get that $\widetilde{f}|_{\widetilde{V}*}(\widetilde{h}_*(\mathcal{L})|_{\widetilde{V}})$ belongs to $D^+_{loc}(V)$. To show that it is constructible it is enough that a stalk at any point $y \in V$ is constructible (complex *with bounded cohomology being finite dimensional vector spaces). By base change we get that*

$$[(\widetilde{f}|_{\widetilde{V}})_*((\widetilde{h}_*\mathcal{L})|_{\widetilde{V}})]_y = (\widetilde{f}_{\widetilde{f}^{-1}(y)})_*((\widetilde{h}_*\mathcal{L})|_{\widetilde{f}^{-1}(y)}).$$

Since dimension of a fiber $\widetilde{f}^{-1}(y)$ is equal to $dim(\widetilde{X} - dim(V) = dim(X) - dim(Y) < dim(X)$ as $dim(Y) > 0$ and, since we proceed by induction, we see that $(\widetilde{f}_{\widetilde{f}^{-1}(y)})_*((\widetilde{h}_*\mathcal{L})|_{\widetilde{f}^{-1}(y)})$. Define $W := Y \setminus V$ and $\widetilde{W} := \widetilde{f}^{-1}(W)$. We just showed that $(f_*h_*\mathcal{L})|_V$ is constructible and, by 0.0.4, it is enough to check that $(f_*h_*\mathcal{L})|_W$ is constructible (let me remind you that the goal of this last step is to show that $f_*h_*\mathcal{L}$ is constructible, and, by step (B) we know that $f_*h_*\mathcal{L}$ being constructible implies f_*F is constructible. The goal of this note is to see constructibility of f_*F). Again, we have

$$(f_*h_*\mathcal{L})|_W = (\widetilde{f}_*\widetilde{h}_*\mathcal{L})|_W = (\widetilde{f}|_{\widetilde{W}})_*[(\widetilde{h}_*\mathcal{L})|_{\widetilde{W}}],$$

the first equality by naturality, the second by proper basechange. We have $dim(\widetilde{W}) < dim(\widetilde{X})$, so we are done by induction again.

0.2 $i^!$ and $R\underline{Hom}$ preserve constructibility

We state two Corollaries of our main result 0.1.1.

Corollary 0.2.1. *Say we have a closed embedding $i : Z \to X$ and a complementary embedding $j : U \to X$. Take $F \in D^b_c$. We can look at a distinguished triangle*

$$i_*i^!F \to F \to j_*j^*F.$$

The last and middle terms are constructible and so is the first. As i is a closed embedding we have $i^*i_*i^!F = i^!F$. Clearly $i^*i_*i^!F$ is constructible and so is $i^!F$

Corollary 0.2.2. *Let $F, G \in D^b_c(X)$. Then $R\underline{Hom}(F, G)$ is constructible again.*

Proof. We proceed by Noetherian induction on X. Choose smooth, connected and open $j : U \to X$ such that $F|_U \in D^b_{loc}(U)$ and let $i : Z \to X$ be a complementary embedding. Now look at a distinguished triangle

$$i_*R\underline{Hom}(i^*F, i^!G) \to R\underline{Hom}(F, G) \to j_*R\underline{Hom}(j^*F, j^*G).$$

Now, $i^!G$ is constructible by Corollary 0.2.1, i^*F is constructible by 0.0.17, therefore $R\underline{Hom}(i^*F, i^!G)$ is constructible by induction, and $R\underline{Hom}(j^*F, j^*G)$ is constructible (even local system) because both j^*F, j^*G are local systems and local systems are closed under $R\underline{Hom}$. Thus the middle term is constructible as well. ∎