6. Constructible sheaves I

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Abstract

In this talk, we will first introduce stratifications and filtrations by smooth varieties, with a list of examples. Then we will define constructible sheaves, the category $D_c^b(X)$, and study some of their properties.

1 Stratifications and filtrations by smooth varieties

Recall that our varieties are defined over $\mathbb{C}$.

Definition 1.1. *Let X be a variety, and let $(X_s)_{s\in\mathscr{S}}$ be a finite collection of disjoint, smooth, connected, and locally closed subvarieties such that $X = \sqcup_{s\in\mathscr{S}} X_s$. Then*

*(1) if we can order $\mathscr{S} = \{s_1, ..., s_n\}$ so that $X_{s_1} \sqcup ... \sqcup X_{s_i}$ is closed for each i, then we say $(X_s)_{s\in\mathscr{S}}$ is a **filtration of X by smooth varieties**.*

*(2) if for any $s, t \in \mathscr{S}$, $\overline{X_s} \cap X_t$ is either empty or X_t, then we say $(X_s)_{s\in\mathscr{S}}$ is a (algebraic) **stratification** of X.*

*The subvarieties X_s are called **strata**.*

Remark. *Since X is Noetherian, it always admits a filtration by smooth varieties: we can first take a smooth connected open subset of X, then apply Noetherian induction to its complement (By **Noetherian induction** we mean the principle that in order to show a property P holds for a noetherian topological space X, we may assume that all proper closed subsets of X satisfies P).*

Remark. *If $(X_s)_{s\in\mathscr{S}}$ is a stratification, we define the **closure partial order** on $\mathscr{S}$ by*

$$t \le s \iff X_t \subset \overline{X_s}$$

Every stratification is automatically a filtration by smooth varieties by choosing any total order on $\mathscr{S}$ that refines the closure partial order.

Example 1.1 (trivial stratification)**.** *If X is a smooth and connected variety, then we have the **trivial stratification** on X, defined in the obvious way.*

Example 1.2 (cell decomposition of $\mathbb{CP}^N$)**.** *We have a stratification of $\mathbb{CP}^N$ by*

$$\mathbb{CP}^N = \mathbb{C}^N \sqcup \mathbb{C}^{N-1} \sqcup ... \sqcup \mathbb{C} \sqcup \{pt\}$$

where $m \le n \iff \mathbb{C}^m \subset \overline{\mathbb{C}^n}$ for $m, n \le N$.

Example 1.3 (cell decomposition for Grassmannians)**.** *(See [AF23, Chapter9] for more details) Denote by $G(d, n)$ the set of $d-$dimensional subspaces of $\mathbb{C}^n$. Recall that $G(d, n)$ is a projective variety of dimension $d \times (n-d)$. We will define a stratification on $G(d, n)$. Fix a **flag** $E_1 \subset E_2 \subset ... \subset E_n = \mathbb{C}^n$ with dim $E_q = q$. Given a partition $\lambda = (n-d \ge \lambda_1 \ge \lambda_2 \ge ... \ge \lambda_d \ge 0)$, the **Schubert cell** Ω°_λ is the set of subspaces*

$$\{F \subset \mathbb{C}^n \mid dim(F \cap E_q) = k \text{ for } q \in [n-d+k-\lambda_k, n-d+k-\lambda_{k+1}], k = 0, .., d\} \quad (1)$$

where we set $\lambda_0 = n-d$, $\lambda_{d+1} = 0$.

The Schubert cells Ω_λ° are obviously disjoint for distinct λ's, and we have

$$G(d,n) = \bigsqcup_{\lambda} \Omega_\lambda^\circ. \tag{2}$$

*It is also not hard to check that these Schubert cells are connected and smooth (they are in fact isomorphic to $\mathbb{C}^{n(n-d)-\sum \lambda_i}$, see the remark below). The closures $\Omega_\lambda := \overline{\Omega_\lambda^\circ}$ are called **Schubert varieties**. They are described by replacing equalities by inequalities in the dimension conditions:*

$$\Omega_\lambda := \overline{\Omega_\lambda^\circ} = \{F \subset \mathbb{C}^n \mid dim(F \cap E_{n-d+k-\lambda_k}) \geq k \text{ for }, k = 1, .., d\} \tag{3}$$

*Next, we illustrate the closure partial order using the **Young diagrams**. Let λ be a partition, say $\lambda = (5, 3, 1, 1)$, then the corresponding young diagram is given by*

1	2	3	4	5
1	2	3		
1				
1				

The closure partial order is given by $\mu \leq \lambda \iff$ the Young diagram of μ covers λ. Then the condition (3) translates to

$$\Omega_\lambda = \bigcup_{\mu \leq \lambda} \Omega_\mu^\circ \tag{4}$$

So we get a stratification on $G(d, n)$.

Example 1.4 (the case of $G(2, 4)$)**.** *We look more closely at the simple case $G(2, 4)$. There are in total 6 partitions, $(2, 2), (2, 1), (2, 0), (1, 1), (1, 0), (0, 0)$ whose Young diagrams are*

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \; \\ \hline \end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \; & \; \\ \hline \end{array}$ $\begin{array}{|c|c|}\hline 1 & \; \\ \hline 1 & \; \\ \hline \end{array}$ $\begin{array}{|c|c|}\hline 1 & \; \\ \hline \; & \; \\ \hline \end{array}$ $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline \end{array}$ *respectively.*

The corresponding Schubert cells can be represented in matrix forms as

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ * & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 0 \\ * & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ * & * \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 0 \\ 0 & 1 \\ * & * \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{pmatrix} \tag{5}$$

Remark. *In general, we have a bijection between partitions $\lambda = (n - d \geq \lambda_1 \geq ... \geq \lambda_d \geq 0)$ with subsets $I = \{i_1 < ... < i_d\} \subset \{1, ..., n\}$ of size d. The bijection is given by*

$$i_k = k + \lambda_{d+1-k} \tag{6}$$

If we choose the standard basis $e_1, ..., e_n$ and fix the flag (as we already did in the example of G(d,n)) $< e_n > \subset < e_{n-1}, e_n > \subset ... \subset < e_1, ...e_n >= \mathbb{C}^n$, then we have

$$\Omega_\lambda^\circ = B^- \cdot F_\lambda \tag{7}$$

where F_λ is the subspace spanned by $\{e_i \mid i \in I\}$ and $B^- \subset GL_n$ is the set of lower triangular matrices(!). This exactly tells us what (5) will look like in general.

Example 1.5 (Bruhat decomposition)**.** *(See, for example [Hum12] for background on reductive groups) Let G be a complex reductive group. We fix a pair $T \subset B$ where T is a maximal torus and B is a Borel subgroup containing T. As an example to keep in mind, we may take G to be GL_n, T to be invertible diagonal matrices, and B to be invertible lower triangular matrices. Then we have the **Bruhat decomposition** for G:*

$$G = \bigsqcup_{w \in W} BwB \tag{8}$$

*where $W = N(T)/T$ is the **Weyl group** of G. Passing to the flag variety G/B, we have*

$$G/B = \bigsqcup_{w \in W} BwB/B \tag{9}$$

*More generally, we say a closed subgroup $P \subset G$ is **parabolic** if $B \subset P$ for some Borel subgroup. Equivalently this means G/P is a projective variety. Then from (9) we get a decomposition for G/P:*

$$G/P = \bigsqcup_{w \in W/W_P} BwP/P \tag{10}$$

where W_P is a subgroup of W determined by P. In fact, (9) (10) are both stratifications (the strata are isomorphic to $\mathbb{C}^{l(w)}$ for some number $l(w)$ which we will not define here).

We can connect this to the example of Grassmannians. Take $G = GL_n$, and recall that it has a transitive left action on $G(d,n)$. Choose a point $p = \begin{pmatrix} 0 \\ I_d \end{pmatrix}$. Its stabilizer is the parabolic subgroup

$$P = \left[\begin{array}{c|c} A & 0 \\ \hline C & D \end{array}\right]$$

where A and D are of size $(n-d)\times(n-d)$ and $d \times d$ respectively. We have $G/P \cong G(d,n)$ as varieties, and we can check that (2) and (10) give rise to the same stratification. For example, if $d = 2$ and $n = 4$, then $W \cong S_4$ and $W_P \cong S_2 \times S_2$, so we have exactly 6 strata on both sides.

Remark. *If the strata of a stratification are all affine varieties, then we say the stratification is an **affine paving**. So far the examples are all affine pavings, except the first one.*

Example 1.6 (action by an algebraic group)**.** *Let G be a connected algebraic group acting on a variety X. Then every orbit is a smooth, connected, locally closed subvariety whose boundary is a union of orbits of lower dimension. Therefore we have a stratification on X. All previous examples are special cases of this, except probably the first one.*

Example 1.7 (Whitney umbrella)**.** *As an example of a stratification that does not come from an algebraic group action, consider $X = V(x^2 - zy^2) \subset \mathbb{C}^3$. The singular locus of X is given by the $z-$axis $\{x = y = 0\}$, and we have a stratification $X = X_{smooth} \sqcup X_{sing}$. But X_{sing} cannot be the orbit of an algebraic group action since the local ring $\mathcal{O}_{X,origin}$ is different from other points on the $z-$axis (compare the completion of the local rings). This example shows that 'equisingularity' along strata may fail.*

Example 1.8 (filtration by smooth varieties $\not\Rightarrow$ stratification)**.** *Let $[x : y : z]$ be the homogeneous coordinates on $\mathbb{CP}^2$. Then the subvarieties $V(y)$, $V(z) \cap D_(y)$, and $D(y) \cap D(z)$ constitute a filtration of $\mathbb{CP}^2$ by smooth varieties. But this is not a stratification as $\overline{V(z) \cap D(y)} = V(z)$*

Definition 1.2. *Let $\mathscr{S}$ and $\mathscr{T}$ be two filtrations of X by smooth varieties. Then $\mathscr{S}$ is a **refinement** of $\mathscr{T}$ if each stratum of $\mathscr{S}$ is contained in a stratum of $\mathscr{T}$.*

Despite the last example, the following should not be surprising.

Lemma 1.1. *Let X be a variety. Then*

(1) Any filtration of X by smooth varieties admits a refinement that is a stratification.

(2) Any two stratifications of X admits a common refinement.

(3) Let $Y \subset X$ be a locally closed subvariety. Then there exists a stratification of X such that Y is a union of strata.

For example, we can refine Example 1.8 to the stratification given by $V(y)\cap V(z)$, $V(y)\cap D(z)$, $V(z)\cap D(y)$, and $D(y) \cap D(z)$.

2 Constructible sheaves

Now we come to the important concept of constructibility.

Definition 2.1. *Let X be a variety, and let $(X_s)_{\mathscr{S}}$ be a stratification of X.*

(1) A sheaf $\mathscr{F} \in Sh(X)$ is said to be ***constructible with respect to*** *$\mathscr{S}$ if for each $s \in \mathscr{S}$, the restriction $\mathscr{F}|_{X_s}$ is a local system of finite type. A sheaf $\mathscr{F}$ is* ***constructible*** *if it is constructible with respect to some stratification of X.*

(2) We say an object $\mathcal{F} \in D^b(X)$ is ***constructible with respect to*** *$\mathscr{S}$ (or just* ***constructible****) if each cohomology sheaf $H^k(\mathcal{F})$ has the same property. The full subcategory of $D^b(X)$ consisting of such objects is denoted by $D^b_{\mathscr{S}}(X)$ (or $D^b_c(X)$)*

Remark. *It is clear that an object $\mathcal{F}$ is constructible with respect to $\mathscr{S}$ if for each X_s we have*

$$\mathcal{F}|_{X_s} \in D^b_{locf}(X_s) \tag{11}$$

where $D^b_{locf}(X_s)$ is the full subcategory of $D^b(X_s)$ consisting of objects $\mathcal{G}$ with $H^i(\mathcal{G}) \in Loc^{ft}(X_s)$ for all $i \in \mathbb{Z}$.

Lemma 2.1. *Let X be a variety.*

(1) For any stratification $\mathscr{S}$ of X, the category $D^b_{\mathscr{S}}(X)$ is a full triangulated subcategory of $D^b(X)$.

(2) The category $D^b_c(X)$ is also a full triangulated subcategory of $D^b(X)$.

We will show(in this and subsequent talks) that various sheaf operations preserve constructibility.

Proposition 2.2. *Let $f : X \to Y$ be a morphism of varieties. Then for any $\mathcal{F} \in D^b_c(X)$, we have $f^*\mathcal{F} \in D^b_c(Y)$.*

Proof. Suppose $\mathcal{F}$ is constructible with respect to $(Y_t)_{t\in\mathscr{T}}$. By Lemma 1.1 we can choose a stratification $(X_s)_{s\in\mathscr{S}}$ of X such that each preimage $f^{-1}(Y_t)$ is a union of strata. By assumption $\mathcal{F}|_{Y_t}$ is a bounded complex with locally constant cohomology sheaves of finite type, so the same is true for $f^*\mathcal{F}|_{f^{-1}(Y_t)}$, and hence is true for each $f^*\mathcal{F}|_{X_s}$. Therefore $f^*\mathcal{F}$ is constructible. □

Proposition 2.3. *Let $h : Y \to X$ be an inclusion of a locally closed subvariety. For any $\mathcal{F} \in D^b_c(X)$, we have $h_!\mathcal{F} \in D^b_c(X)$. In particular, if Y is closed, then $h_*\mathcal{F} \in D^b_c(X)$.*

Proof. We may assume $\mathcal{F}$ is constructible with respect to some stratification $\mathscr{S}$ for which Y is a union of strata. If $X_s \subset Y$, then $(h_!\mathcal{F})|_{X_s} \cong \mathcal{F}|_{X_s}$, and this lies in $D^b_{locf}(X)$ by assumption. If $X_s \nsubseteq Y$, then $(h_!\mathcal{F})|_{X_s} = 0$ since $X_s \cap Y = \varnothing$. □

Proposition 2.4. *Let $\mathscr{S}$ be a stratification of X. If $\mathcal{F}$ and $\mathcal{G}$ are objects in $D^b_{\mathscr{S}}(X)$, then so is $\mathcal{F}\otimes^L\mathcal{G}$. In particular, if $\mathcal{F}$ and $\mathcal{G}$ are objects in $D^b_c(X)$, then so is $\mathcal{F}\otimes^L\mathcal{G}$.*

Proof. Since $(\mathcal{F}\otimes^L\mathcal{G})|_{X_s} \cong (\mathcal{F}|_{X_s})\otimes^L(\mathcal{G}|_{X_s})$, it suffices to prove the next lemma. □

Lemma 2.5. *Let X be a smooth and connected variety. If $\mathcal{F}$ and $\mathcal{G}$ are objects in $D^b_{locf}(X)$, then so is $\mathcal{F}\otimes^L\mathcal{G}$.*

Proof. Because $D^b_{locf}(X)$ is a full triangulated subcategory, we may use truncation and induction on the number of nonzero cohomology sheaves of $\mathcal{F}$. So we can reduce to the case where $\mathcal{F}$ is just a sheaf (i.e., a local system of finite type). By choosing an open neighborhood of each point where $\mathcal{F}$ is a constant sheaf, we may assume $\mathcal{F} = \underline{M}$ for some $\mathbb{C}-$vector space M. In particular, M is free, and $\otimes^L$ is just the usual tensor product. So our result follows from the isomorphism ([Ach21, Proposition1.4.4]) $\underline{\mathbb{C}} \otimes \mathcal{G} \tilde{\to} \mathcal{G}$. □

Finally we recall some results from algebraic geometry. Recall from Talk 2 or [Ach21, Section1.2] that a continuous map $f : X \to Y$ is **proper** if it is universally closed. If X and Y are locally compact, then the following conditions are all equivalent:

(1) f is proper

(2) If $K \subset Y$ is compact, then the set $f^{-1}(K)$ is also compact

(3) f is a closed map, and $f^{-1}(y)$ is compact for every point $y \in Y$.

We also have the notion of a **proper morphism** between schemes. In the case of varieties, this will imply our definition above of a proper map between locally compact spaces.

Theorem 2.6 (Nagata's compactification theorem)**.** *Let $X \to Y$ be a morphism of varieties. There exists a variety $\tilde{X}$, an open embedding $j : X \to \tilde{X}$, and a proper morphism $\tilde{f} : \tilde{X} \to Y$ such that the following is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle j} & \nearrow{\scriptstyle \tilde{f}} & \\ \tilde{X} & & \end{array}$$

See [Ach21, Section2.1] for more discussion and references for this and the Ehresmann's fibration theorem below.

Definition 2.2. *Let X, Y be smooth manifolds. A differentiable map $f : X \to Y$ is said to be a* ***differentiable locally trivial fibration*** *(with fiber F) if for each $y \in Y$, there is a neighborhood $y \in U$ and a diffeomorphism $b : f^{-1}(U) \tilde{\to} F \times U$ such that $pr_2 \circ b = f|_{f^{-1}(U)}$, where pr_2 is the projection from $F \times U$ to U.*

Theorem 2.7 (Ehresmann's fibration theorem)**.** *Let $f : X \to Y$ be a smooth, surjective, proper morphism of smooth varieties. Then f is a differentiable locally trivial fibration.*

References

[Hum12] James E Humphreys. *Linear algebraic groups*. Vol. 21. Springer Science & Business Media, 2012.

[Ach21] Pramod N Achar. *Perverse sheaves and applications to representation theory*. Vol. 258. American Mathematical Soc., 2021.

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