TOWARDS CONSTRUCTIBLE SHEAVES: !-PULLBACK OF SMOOTH MORPHISMS

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In the last lectures we have introduced the !-pullback as the right adjoint of the !-pushforward. Working with this description of the !-pullback is in general not an easy task, however in this lecture we present a simple description of the !-pullback for smooth morphisms of varieties. Additionally, we prove that the smooth pull-back commutes with all sheaf operations. This lecture is part of a series of lectures of a seminar on Perverse Sheaves taking place at the University of Bonn and is based on [Ach21] and suggestions made by Dr. Stefan Dawydiak.

Note: We saw in Lecture 3 that in order to be able to talk about $f^!\colon D^+(Y) \longrightarrow D^+(X)$ for a continuous morphism $f\colon X \longrightarrow Y$, we need to require X, Y to be locally compact and ${}^\circ f_!$ to have finite cohomological dimension. In this lecture we consider continuous morphisms between varieties, which were proved to be locally compact and to have finite c-soft dimension in the analytic topology. In Lecture 3 was also shown that ${}^\circ f_!$ has finite cohomological dimension if Y is locally compact and X has finite c-soft dimension. Thus, in the context of this lecture we may talk about the !-pullback $f^!$ contructed as the right adjoint of the !-pushforward.

Theorem 0.1. *Let $f\colon X \longrightarrow Y$ be a smooth morphism of varieties of relative dimension d and let $\mathcal{F} \in D^b(Y)$, then we have an isomorphism $f^!\mathcal{F} \simeq f^*\mathcal{F}[2d]$.*

Proof. We prove here only the special case: $Y = pt$ and $\mathcal{F} \simeq \underline{\mathbb{C}}_{pt}$. Hence, X is a smooth variety of dimension $\dim(X) = d$. For a more general proof look at Theorem 1.5.11. in [Ach21].

Step 1: We first show that $f^!\underline{\mathbb{C}}_{pt}$ is a rank 1 local system on X in degree $-2d$.
This will be done by proving that for any $U \subset X$ small enough, we have $R\Gamma(U, f^!\underline{\mathbb{C}}_{pt}|_U) \simeq \underline{\mathbb{C}}[2d]$ and applying Theorem 1.7 of Lecture 3. By adjunction and using $j^* = j^!$, we have $R\Gamma(U, f^!\underline{\mathbb{C}}_{pt}|_U) \simeq \mathrm{Hom}(\underline{\mathbb{C}}_U, j^! f^! \underline{\mathbb{C}}_{pt}) \simeq \mathrm{Hom}((f \circ j)_! \underline{\mathbb{C}}_U, \underline{\mathbb{C}}_{pt})$.
Since $f \circ j\colon U \longrightarrow \{pt\}$ is the structure morphism of U, we have ${}^\circ(f \circ j)_! \underline{\mathbb{C}}_U = \Gamma_c(\underline{\mathbb{C}}_U)$, so $(f \circ j)_! \underline{\mathbb{C}}_U = R\Gamma_c(\underline{\mathbb{C}}_U) = H^*_c(U, \mathbb{C})$. Choose U to be small enough such that $U \simeq \mathbb{R}^{2d}$ and recall that $H^i_c(\mathbb{R}^{2d}, \mathbb{C}) = \varinjlim_{K \subset \mathbb{R}^{2d} \text{ compact}} H^i(\mathbb{R}^{2d}, \mathbb{R}^{2d} \backslash K)$. Since every compact $K \subset \mathbb{R}^{2d}$ is contained in a closed ball of radius r, $\overline{B(r)}$, we have $H^i_c(\mathbb{R}^{2d}, \mathbb{C}) = \varinjlim_{r>0} H^i(\mathbb{R}^{2d}, \mathbb{R}^{2d} \backslash \overline{B(r)}) \simeq H^i(\mathbb{R}^{2d}, \mathbb{R}^{2d} \backslash \{*\})$.
Thus, $R\Gamma(U, f^!\underline{\mathbb{C}}_{pt}|_U) \simeq \mathrm{Hom}(H^*_c(U, \mathbb{C}), \underline{\mathbb{C}}_{pt}) \simeq \mathrm{Hom}(\underline{\mathbb{C}}_{pt}[-2d], \underline{\mathbb{C}}_{pt}) \simeq \underline{\mathbb{C}}[2d]$.

Step 2: FACT (Poincaré duality): Let X be a smooth variety and let $\mathcal{L}$ be a local system on X. Hence, we have following isomorphism

$$(\mathbf{H}_c^k(X, \mathcal{L}^*))^* \simeq \mathbf{H}^{2d-k}(X, \mathcal{L}),$$

where $\mathcal{L}^*$ is the dual of $\mathcal{L}$ and, $\mathbf{H}^* := H^*(R\Gamma(-))$ and $\mathbf{H}_c^* := H^*(R\Gamma_c(-))$ denote the hypercohomology and the hypercohomology with compact support respectively. Note that in the case $\mathcal{L} = \underline{\mathbb{C}}_X$ we obtain the usual Poincaré duality.

Step 3: Finally, we want to use steps 1 and 2 to prove that $f^!\underline{\mathbb{C}}_{pt} \simeq f^*\underline{\mathbb{C}}_{pt}[2d]$.
Since rank 1 local systems on X form a full subcategory of $\mathrm{Loc}(X)$, by the Yoneda Lemma it suffices to prove that

$$RHom_{D(X)}(\mathcal{L}, f^!\underline{\mathbb{C}}_{pt}) \simeq RHom_{D(X)}(\mathcal{L}, f^*\underline{\mathbb{C}}_{pt}[2d]) \tag{0.1}$$

holds for any rank 1 local system $\mathcal{L}$ on X. Let us first consider the left hand side of (0.1). Via adjunction we obtain:

$$RHom_{D(X)}(\mathcal{L}, f^!\underline{\mathbb{C}}_{pt}) \simeq RHom_{D(pt)}(f_!\mathcal{L}, \underline{\mathbb{C}}_{pt}) \simeq RHom_{D(pt)}(R\Gamma_c(\mathcal{L}), \underline{\mathbb{C}}_{pt}) \simeq R\Gamma_c(\mathcal{L})^*.$$

Let us consider now the right-hand-side of (0.1). Since f is the structure morphism of X, we have $f^*\underline{\mathbb{C}}_{pt} \simeq \underline{\mathbb{C}}_X$. Thus,

$$RHom_{D(pt)}(\mathcal{L}, f^*\underline{\mathbb{C}}_{pt}[2d]) \simeq RHom_{D(pt)}(\mathcal{L}, \underline{\mathbb{C}}_X[2d]) \simeq R\Gamma(\mathcal{L}^*[2d]).$$

After taking cohomology on both sides of (0.1), we obtain $H^i(RHom_{D(X)}(\mathcal{L}, f^!\underline{\mathbb{C}}_{pt})) = \mathbf{H}_c^i(X, \mathcal{L})^*$ and $H^i(RHom_{D(X)}(\mathcal{L}, f^*\underline{\mathbb{C}}_{pt}[2d])) = \mathbf{H}^{2d-i}(X, \mathcal{L}^*)$.

Finally, since in (0.1) both complexes are complexes of vector spaces, they are equal if they have the same cohomology. So Poincaré duality yields the claimed result. □

Definition 0.2 (Dualizing complex)**.** Let X be a variety and let $a_X \colon X \longrightarrow \{pt\}$ be the constant morphism. We define the dualizing complex as:

$$\omega_X := a_X^!\underline{\mathbb{C}}_{pt}.$$

Remark 0.3. Note that this definition works as well for locally compact spaces of finite c-soft dimension.

Proposition 0.4. *Let $f \colon X \longrightarrow Y$ be a morphism of varieties. Then, $f^!\omega_Y \simeq \omega_X$.*

Proof. Follows from functoriality of the !-pullback. □

Corollary 0.5. *Let X be a smooth variety of dimension d. Then, we have $\omega_X \simeq \underline{\mathbb{C}}_X[2d]$.*

Proof. Theorem 0.1 yields $\omega_X \simeq a_X^*\underline{\mathbb{C}}_{pt}[2d] \simeq \underline{\mathbb{C}}_X[2d]$. □

Theorem 0.6 (Smooth pull-back commutes with all sheaf operations.)**.** *Let $f\colon X \longrightarrow Y$ be a smooth morphism of varieties of relative dimension d and let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a Cartesian diagram. Then, f' is smooth of relative dimension d and for $\mathcal{F}, \mathcal{G} \in D^b(Y)$ and $\mathcal{H} \in D^b(Y')$, we have following natural isomorphisms:

$$f^*(\mathcal{F} \otimes \mathcal{G}) \simeq f^*\mathcal{F} \otimes f^*\mathcal{G},$$

$$g'^* f^* \mathcal{F} \simeq f'^* g^* \mathcal{F},$$

$$f^* g_! \mathcal{H} \simeq g'_! f'^* \mathcal{H},$$

$$f^* R\mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \simeq R\mathscr{H}\mathrm{om}(f^*\mathcal{F}, f^*\mathcal{G}),$$

$$g'^! f^* \mathcal{F} \simeq f'^* g^! \mathcal{F},$$

$$f^* g_* \mathcal{H} \simeq g'_* f'^* \mathcal{H}.$$

Proof. The statement that being smooth of relative dimension d is stable under base change has been proved in Lecture 4 (part 2).
The first isomorphism has been proved in the second lecture (part 2) and does not require smoothness.
The second isomorphism follows from following isomorphisms proved in Lecture 2 (Proposition 1.25.): $g'^* f^* \mathcal{F} \simeq (f \circ g^{'})^* \mathcal{F}$ and $f'^* g^* \mathcal{F} \simeq (g \circ f')^* \mathcal{F}$, which do not require smoothness.
The third isomorphism was already proved in Lecture 3 (Theorem 4.3.) without the assumption of smoothness.
In Lecture 3 was shown (Proposition 5.8.) that $f^! R\mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{G}[-2d]) \simeq R\mathscr{H}\mathrm{om}(f^*\mathcal{F}, f^!\mathcal{G}[-2d])$ holds, hence Theorem 0.1 yields following isomorphism:

$$R\mathscr{H}\mathrm{om}(f^*\mathcal{F}, f^*\mathcal{G}) \simeq f^* R\mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{G}[-2d])[2d] \simeq f^* R\mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{G}).$$

Now, Theorem 0.1 allows us to write $f^* \simeq f^![-2d]$. Hence, since $g'^! f^! \mathcal{F} \simeq (f \circ g')^! \mathcal{F}$ by Proposition 5.6. of Lecture 3, we obtain

$$g'^! f^* \mathcal{F} \overset{\text{Thm 0.1}}{\simeq} g'^! f^! \mathcal{F}[-2d] \simeq f'^! g^! \mathcal{F}[-2d] \overset{\text{Thm 0.1}}{\simeq} f'^* g^! \mathcal{F}.$$

Finally, in Lecture 3 (Proposition 5.7.) was proved that $f^! g_* \mathcal{H} \simeq g'_* f'^! \mathcal{H}$ for $\mathcal{H} \in D^+(Y')$. Hence, Theorem 0.1 yields:

$$f^* g_* \mathcal{H}[2d] \overset{\text{Thm 0.1}}{\simeq} f^! g_* \mathcal{H} \simeq g'_* f'^! \mathcal{H} \overset{\text{Thm 0.1}}{\simeq} g'_* f'^* \mathcal{H}[2d].$$

□

Theorem 0.7. *Let X be a smooth, equidimensional variety and let Y be a smooth locally closed equidimensional subvariety. Let $h\colon Y \hookrightarrow X$ be the inclusion map, and let $d = \dim X - \dim Y$. For $\mathcal{L} \in \mathrm{Loc}(X)$, there is a natural isomorphism $h^!\mathcal{L} \simeq h^*\mathcal{L}[-2d]$.*

Proof. Let $j\colon D \hookrightarrow X$ be an open disk and consider following Cartesian diagram:

$$\begin{array}{ccc} V & \xrightarrow{h'} & D \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{h} & X \end{array}$$

Note that being open immersion is stable under base change, i.e. if j is an open immersion, so is its base change j'. Hence, $j^* = j^!$ and $j'^* = j'^!$ and we have:

$$h'^!(\mathcal{L}|_D) = h'^! j^! \mathcal{L} \simeq j'^! h^! \mathcal{L} = (h^!\mathcal{L})|_V.$$

Since $\mathcal{L}$ is a local system, by choosing D small enough we have that $\mathcal{L}|_D \simeq \underline{M}_D$ is a constant sheaf. Let $d_X = \dim(X)$, thus Theorem 0.1 yields

$$\underline{M}_D \simeq \underline{M}_X|_D \simeq (a_X^* \underline{M}_{pt})|_D \simeq (a_X^! \underline{M}_{pt}[-2d_X])|_D, \tag{0.2}$$

where $a_X\colon X \longrightarrow \{pt\}$ is the structure morphism of X. Note that $a_X \circ h\colon Y \longrightarrow \{pt\}$ is the structure morphism of Y, hence $h'^! j^! a_X^! \underline{M}_{pt} \simeq j'^! h^! a_X^! \underline{M}_{pt} = j'^! a_Y^! \underline{M}_{pt}$ by Theorem 0.6. Additionally, as has been shown in (0.2) we have $j'^! a_Y^! \underline{M}_{pt} \simeq \underline{M}_Y|_V[2d_Y]$. Thus,

$$(h^!\mathcal{L})|_V = j'^! h^! \mathcal{L} \simeq h'^! j^! \mathcal{L} \simeq h'^! j^! a_X^! \underline{M}_{pt}[-2d_X] \simeq j'^! a_Y^! \underline{M}_{pt}[-2d_X] \simeq \underline{M}_V[-2d] \simeq h^*\mathcal{L}|_V[-2d].$$

We can cover Y with such open subsets V and the restriction maps commute with the above isomorphism, hence the above isomorphism implies $h^!\mathcal{L} \simeq h^*\mathcal{L}[-2d]$. □

Example 0.8. *Consider following inclusion $i\colon Y = \{0\} \hookrightarrow X = \mathbb{C}$. Hence, as was shown in Lecture 2, we obtain $i^! \underline{\mathbb{C}}_X \simeq i^! \omega_X[-2] \simeq \omega_{pt}[-2] \simeq \underline{\mathbb{C}}_{pt}[-2]$.*

References

[Ach21] P.N. Achar. *Perverse sheaves and applications to representation theory.* Mathematical surveys and monographs. American mathematical society, 2021.