Towards Constructible Sheaves I

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Contents

Recall that in last talk, we proved that for each cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

and for each $\mathcal{F} \in D^+(X)$, we have a commutative diagram

$$\begin{array}{ccc} g^* f_! \mathcal{F} & \longrightarrow & f'_!(g')^* \mathcal{F} \\ \downarrow & & \downarrow \\ g^* f_* \mathcal{F} & \longrightarrow & f'_* g'_* \mathcal{F} \end{array}$$

The two horizontal map are called **base change morphism**. We also have seen that

Proposition 0.1. *If f is proper, then for each $\mathcal{F} \in D^+(X)$,*

$$g^* f_* \mathcal{F} \to f'_*(g')^* \mathcal{F}$$

is an isomorphism.

Today, we will give more conditions for this base change morphism to be an isomorphism.

1 Base change for products

Let $f : X \to X'$ be a continuous map. Consider the following cartesian square

$$\begin{array}{ccc} X \times Y & \xrightarrow{\mathrm{pr}_1} & X \\ \downarrow{\scriptstyle f' = f \times \mathrm{Id}} & & \downarrow{\scriptstyle f} \\ X' \times Y & \xrightarrow{\mathrm{pr}_1} & X' \end{array}$$

Proposition 1.1. *Let Y be a locally contractible space, i.e., the collection of contractible space form a basis for the topology of Y. For $\mathscr{F} \in D^+(X)$, the base change map $\mathrm{pr}_1^* f_* \mathcal{F} \to f'_* \mathrm{pr}_1^* \mathcal{F}$ is an isomorphism.*

Proof. We only prove the abelian version, i.e., $\mathrm{pr}_1^* {}^\circ f_* \mathcal{F} \to {}^\circ f'_* \mathrm{pr}_1^* \mathcal{F}$ is an isomoprhism. To prove this, we need the following lemma:

Lemma 1.2. *Let $\mathrm{pr}_1 : X \times Y \to X$ be the projection map. Let $\mathcal{F} \in \mathrm{Sh}(X)$. For any open set $U \subset X$ and any connected open set $V \subset Y$, we have*

$$\mathrm{pr}_1^* \mathcal{F}(U \times V) \cong \mathcal{F}(U).$$

Then for all open set $U \subset X'$ and connected open set $V \subset Y$, we have

$$\mathrm{pr}_1^* {}^\circ f_* \mathcal{F}(U \times V) = {}^\circ f_* \mathcal{F}(U \times V) = \mathcal{F}(f^{-1}(U)),$$

and

$${}^\circ f'_* \mathrm{pr}_1^* \mathcal{F}(U \times V) = \mathrm{pr}_1^* \mathcal{F}(f^{-1}(U) \times V) = \mathcal{F}(f^{-1}(U)).$$

Fact: Locally contractible spaces are locally connected.

Hence the open sets of the form $U \times V$ form a basis for $X' \times Y$ and the two sheaf are isomorphic on this basis. Thus they are isomorphic. ☐

2 Base change for locally trivial fibrations

In this section, we assume that all topological are locally contractible.

Definition 2.1. *A continuous map $f : X \times Y$ is called a* ***locally trivial fibration*** *(with fiber F) if there is an open cover $\{U_\alpha\}$ of Y such that there is a homeomorphism $f^{-1}(U) \to U \times F$ making the following diagram commute:*

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & U \times F \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

Note that $f^{-1}(y)$ is homeomorphic to F for all $y \in Y$, so $f^{-1}(y) \times U$ is homeomorphic to $f^{-1}(U)$ for each open set $U \subset Y$ and point $y \in U$.

Recall the notion of local system: A sheaf $\mathcal{L}$ on a topological space X is called a **local system** if Y admits an open cover $\{U_\alpha\}$ such that $\mathcal{L}|_{U_\alpha}$ is a constant sheaf. We use the notion $D^+_{\mathrm{loc}}(X)$ to denote the subcategory of $D^+(X)$ whose objects are chain complex $\mathcal{F}$ such that $H^k(\mathcal{F}) \in \mathrm{Loc}(X)$ for all $k \in \mathbb{Z}$.

By last talk, we know that if $f : X \to Y$ is a continuous map of locally contractible spaces, then f^* takes local systems on Y to localy systems on X and it induces a functor

$$f^* : D^+_{\mathrm{loc}}(Y) \to D^+_{\mathrm{loc}}(X).$$

Here are more useful fact about $D_{\mathrm{loc}}(X)$.

Fact 1: If $f, g : X \to Y$ are homotopic maps, then the pullback functors $f^*, g^* : D^+_{\mathrm{loc}}(Y) \to D^+_{\mathrm{loc}}(X)$ are isomorphic.

Fact 2: If $f : X \to Y$ is a homotopy equivalence between two locally contractible spaces with homotopy inverse $g : Y \to X$. Then $f^* : D^+_{\mathrm{loc}}(Y) \to D^+_{\mathrm{loc}}(X)$ is an equivalence of categories, and $g^* : D^+_{\mathrm{loc}}(X) \to D^+_{\mathrm{loc}}(Y)$ is an inverse of f^*. This along with Fact 1 show that $D^+(X)$ is a homotopy invariant.

Fact 3: $\mathrm{Loc}(X)$ is closed under extensions, i.e., for each short exact sequence in $\mathrm{Sh}(X)$,$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ with $\mathcal{F}, \mathcal{H} \in \mathrm{Loc}(X)$, then $\mathcal{G} \in \mathrm{Loc}(X)$. Note that this implies that $D_{\mathrm{loc}}(X)$ (resp. $D^+_{\mathrm{loc}}(X)$, resp. $D^-_{\mathrm{loc}}(X)$, resp. $D^b_{\mathrm{loc}}(X)$) is a fully triangulated subcategory of $D(X)$ (resp. $D^+(X)$, resp. $D^-(X)$, resp. $D^b(X)$).

We have the following theorem:

Theorem 2.2. *Let $f : X \to Y$ be a locally trivial fibration.*

(1) For $\mathcal{F} \in D^+_{\mathrm{loc}}(X)$, $f_*\mathcal{F} \in D^+_{\mathrm{loc}}(Y)$.

(2) Consider the following cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let $\mathcal{F} \in D^+_{\mathrm{loc}}(X)$. Then the base change $g^* f_* \mathcal{F} \to f'_*(g')^*\mathcal{F}$ is an isomorphism.

To prove this theorem, we need the following proposition:

Proposition 2.3. *Let $f : X \to X'$ be a continuous map, and let Y be a contractible space. Let $y_0 \in Y$. Let $f' = f \times \mathrm{Id} : X \times Y \to X' \times Y$. Consider the cartesian square*

$$\begin{array}{ccc} X & \xrightarrow{i : x \mapsto (x, y_0)} & X \times Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f' = f \times \mathrm{Id}} \\ X' & \xrightarrow{i' : x' \mapsto (x', y_0)} & X' \times Y \end{array}$$

For $\mathcal{F} \in D^+_{\mathrm{loc}}(X \times Y)$, the base change map $(i')^* f'_* \mathcal{F} \to f_* i^* \mathcal{F}$ is an isomorphism.

Proof of Theorem 2.2. We first assume that $Y' = \{y\}$ for some $y \in Y$. Since f is a locally trivial fibration, there exists a contractible open neighborhood U of y such that $f^{-1}(U)$ is homeomorphic to $f^{-1}(y) \times U$. Consider the following diagram

$$\begin{array}{ccccc} & & \overset{h'}{\frown} & & \\ f^{-1}(y) & \xrightarrow{i'} & f^{-1}(y) \times U = f^{-1}(U) & \xhookrightarrow{j'} & X \\ \downarrow{\scriptstyle f|_{f^{-1}(y)} =: f''} & & \downarrow{\scriptstyle f|_{f^{-1}(U)} =: f'} & & \downarrow{\scriptstyle f} \\ \{y\} & \xrightarrow{i} & \{y\} \times U = U & \xrightarrow{j} & Y \\ & & \underset{h}{\smile} & & \end{array}$$

By Proposition above, we see that for $(j')^*\mathcal{F} \in D^+_{\mathrm{loc}}(f^{-1}(y) \times U)$, the base change map of the left square

$$i^*(f')_*(j')^*\mathcal{F} \to (f'')_*(i')^*(j')^*\mathcal{F} = (f'')_*(h')^*\mathcal{F}$$

is an isomorphism.

By Proposition 1.2.16 from Achar's book, we see that the base change map of the right square

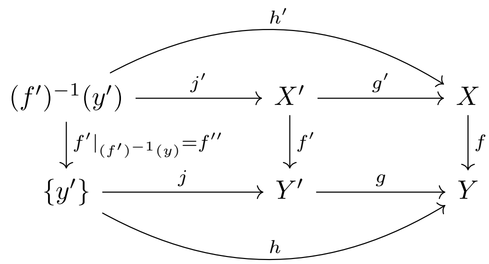
$$j^* f_* \mathcal{F} \to (f')_*(j')^*\mathcal{F}$$

is an isomoprhism and hence

$$i^* j^* f_* \mathcal{F} = h^* f_* \mathcal{F} \to i^*(f')_*(j')^*\mathcal{F}$$

is an isomorphism. Therefore $h^* f_* \mathcal{F} \to (f|_{f^{-1}(y)})_*(h')^*\mathcal{F}$ is an isomorphism.

For the general case, it suffices to prove that $(g^* f_* \mathcal{F})_{y'} \cong (f'_*(g')^*\mathcal{F})_{y'}$ for each $y' \in Y'$. Consider the following diagram



<u>Fact:</u> Locally trivial fibrations are stable under base change.

By special case, we get

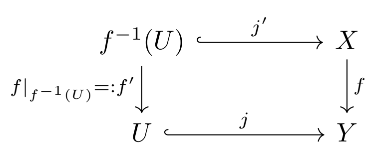
$$h^* f_* \mathcal{F} = (g^* f_* \mathcal{F})_{y'} \to f''_*(h')^*\mathcal{F}$$

and

$$j^* f'_*(g')^*\mathcal{F} = (f'_*(g')^*\mathcal{F})_{y'} \to f''_*(j')^*(g')^*\mathcal{F} = f''_*(h')^*\mathcal{F}$$

are isomorphisms. This proves statement (2). (Note that the reason why we can apply the special case to the whole cartesian square is that $(f')^{-1}(y') \to X$ is an inclusion.)

We now prove statement (1). Let U be a contractible open subset of Y. Choose $y \in U$. Let $j : U \to Y$ be the inclusion. Consider the following cartesian square



Then we have

$$f_* \mathcal{F}|_U = j^* f_* \mathcal{F} \cong (f')_*(j')^*\mathcal{F}. \tag{1}$$

Since U is contractible, the inclusion $i' : f^{-1}(y) \to f^{-1}(U) = f^{-1}(y) \times U$ is a homotopy equivalence with inverse $\mathrm{pr}_1 : f^{-1}(y) \times U \to f^{-1}(y)$. Hence (1) is turned to

$$f_* \mathcal{F}|_U = (f')_* \,\mathrm{pr}_1^*(i')^*(j')^*\mathcal{F} = (f')_* \,\mathrm{pr}_1^*(\mathcal{F}|_{f^{-1}(y)}). \tag{2}$$

Consider the following cartesian square

$$\begin{array}{ccc} f^{-1}(y) \times U = f^{-1}(U) & \xrightarrow{\mathrm{pr}_1} & f^{-1}(y) \\ {\scriptstyle f_{f^{-1}(y)} \times \mathrm{Id} = f|_{f^{-1}(U)} = f'} \downarrow & & \downarrow {\scriptstyle a_{f^{-1}(y)}} \\ \{y\} \times U = U & \xrightarrow{\mathrm{pr}_1 = a_U} & \{y\} = \{pt\} \end{array}$$

By Proposition 1.1, equation (2) is turned into

$$f_*\mathcal{F}|_U = a_U^*(a_{f^{-1}(y)})_*\mathcal{F}|_{f^{-1}(y)} = a_U^* R\Gamma(\mathcal{F}|_{f^{-1}(y)}) = a_U^*(f_*\mathcal{F})_y.$$

The last equation follows the special case we proved above. Note that $H^k(a_U^*(f_*\mathcal{F})_y) = a_U^*(H^k(f_*\mathcal{F})_y)$ since a_U^* is exact. Note that $a_U^*(H^k((f_*\mathcal{F})_y)$ is a constant sheaf. Hence $(f_*\mathcal{F})|_U$ has constant cohomology sheaves. So $f_*\mathcal{F}$ has locally constant cohomology sheaves. This completes the proof. □

Proof of Proposition 2.3. Consider the larger diagram

$$\begin{array}{ccccc} & & \overset{\mathrm{Id}}{\frown} & & \\ X & \xrightarrow{i} & X \times Y & \xrightarrow{\mathrm{pr}_1} & X \\ \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle f'} & & \downarrow {\scriptstyle f} \\ X' & \xrightarrow{i'} & X' \times Y & \xrightarrow{\mathrm{pr}_1} & X' \\ & & \underset{\mathrm{Id}}{\smile} & & \end{array}$$

Since Y is contractible, the projection $\mathrm{pr}_1 : X \times Y \to X$ is an homotopy equivalence. By Fact 2, $\mathrm{pr}_1^* : D^+_{\mathrm{loc}}(X) \to D^+_{\mathrm{loc}}(X \times Y)$ is an equivalence of categories. Therefore $\mathcal{F} = \mathrm{pr}_1^*(\mathcal{G})$ for some $\mathcal{G} \in D^+_{\mathrm{loc}}(X)$. By Proposition 1.1, the base change map

$$\mathrm{pr}_1^* f_*\mathcal{G} \to f'_* \mathrm{pr}_1^* \mathcal{G} = f'_*\mathcal{F}$$

is an isomorphism. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Id}_{X'}^* f_*\mathcal{G} = (i')^* \mathrm{pr}_1^* f_*\mathcal{G} & \xrightarrow{\cong} & (i')^* f'_* \mathrm{pr}_1^* \mathcal{G} = (i')^* f'_*\mathcal{F} \\ {\scriptstyle \cong} \searrow & & \swarrow \\ & f''_* \mathrm{Id}_X^* \mathcal{G} = f''^*_* \mathcal{F} & \end{array}$$

Therefore the base change map

$$(i')^* f'_*\mathcal{F} \to f_* i^* \mathcal{F}$$

is an isomorphism. This completes the proof. □

3 Preliminaries from complex algebraic geometry

We now introduction some notions from complex algebraic geometry and see its connection with the functors introduced before.

A variety is a quasiprojective complex algebraic variety, i.e., a subset of some projective space $\mathbb{P}^n$ that is locally closed in the Zariski topology. Note that there are now two topologies on a variety—the Zariski topology and the analytic topology induced by the analytic topology on $\mathbb{P}^n$. *Sheaves will always be considered with respect to the analytic topology* since the analytic topology is much better suited to applying results from Talk 1-3. The **dimension** of a variety is its algebraic dimension, i.e., dimension as a variety. Smooth varieties are always assumed to be **equidimensional**, i.e., all connected components have the same dimension.

Theorem 3.1. *In the analytic topology, every variety is locally compact, locally contractible, and of finite c-soft dimension.*

Definition 3.2. *Let X be a locally compact space. We say that X has* ***c-soft dimension*** *$\leq n$ if the functor ${}^{\circ}a_{X!}$ has cohomological dimension $\leq n$.*

Fact: X has c-soft dimension $\leq n$ if and only if every sheaf admits a c-soft resolution of length $\leq n$.

Definition 3.3. *A morphism $f : X \to Y$ of varieties is* ***proper*** *if the preimages of compact sets are compact. It is* ***finite*** *if it is proper and quasi-finite.*

Example 3.4. *Every closed immersion are finite. Every polynomial morphisms $\mathbb{C} \to \mathbb{C}$ is finite. Open immersions are quasi-finite and not proper.*