Local systems and proper base change

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Motivation

In this talk we will introduce the notion of a local system. Local systems are particularly easy sheaves, and under favourable circumstances their data is equivalent to a representation of the fundamental group of the underlying space.

We will also get to know proper pushforward, yet another functor between categories of sheaves, and the right adjoint to its derived functor. We will see that properness is a nice property as it will give us several useful base change isomorphisms.

1 Local systems

Definition 1.1. A sheaf $\mathcal{L}$ on a topological space X is called a local system if there is an open cover $\{U_\alpha\}_\alpha$ such that $\mathcal{L}|_{U_\alpha}$ is a constant sheaf for all α. By $\operatorname{Loc} X \subset \operatorname{Sh} X$ we denote the full subcategory of local systems on X. We say a local system $\mathcal{L}$ on X is of finite type if $\mathcal{L}_x$ is a finite-dimensional $\mathbb{C}$-vector space for all $x \in X$. The full subcategory of finite-type local systems on X is denoted $\operatorname{Loc}^{\mathrm{ft}} X$.

Remark 1.2. A few quick observations:

1. Let $f : X \to Y$ be continuous, then f^* takes local systems on Y to local systems on X. This is because f^* takes constant sheaves to constant sheaves.
2. If X is connected and $\mathcal{L}$ is a local system on X, then all stalks of $\mathcal{L}$ are isomorphic.

Remark 1.3. Here is a short reminder of an important property of constant sheaves: If $\underline{M}_X$ is a constant sheaf on a connected, locally connected space X, then the map $\Gamma(X, \underline{M}_X) \to \underline{M}_{X,x}$ is an isomorphism for all $x \in X$, and both can be canonically identified with M.

Proposition 1.4. *Suppose X is connected and locally connected. Then the constant sheaf functor is fully faithful.*

Proof. We have $\operatorname{Hom}(\underline{M}_X, \underline{N}_X) \cong \operatorname{Hom}_{\mathbb{C}}(M, \Gamma(X, \underline{N}_X)) \cong \operatorname{Hom}_{\mathbb{C}}(M, N)$, the former due to adjunction, the latter due to 1.3 □

Proposition 1.5. *If X is locally connected, then $\operatorname{Loc} X$ is an abelian subcategory of $\operatorname{Sh} X$.*

Proof. Let $\phi : \mathcal{L} \to \mathcal{L}'$ be a morphism of local systems. Let $x \in X$ and pick an open connected neighbourhood U of x such that $\mathcal{L}|_U$ and $\mathcal{L}'|_U$ are constant. Because of Proposition 1.4, both $\ker \phi|_U$ and $\operatorname{coker} \phi|_U$ are also constant. □

Example 1.6. Let $X = \mathbb{C} \setminus \{0\}$. We define the sheaf $\mathcal{F} \in \operatorname{Sh} X$ via

$$\mathcal{F}(U) = \{\lambda g \mid \lambda \in \mathbb{C},\, g : U \to \mathbb{C} \text{ holomorphic such that } g(z)^2 = z\}$$

Then $\mathcal{F}$ is a local system, because locally around any $x \in X$ there exists a unique (up to sign) holomorphic square root function g. However $\mathcal{F}$ is not itself a constant sheaf, as it has no nontrivial global sections.

We end this section with a useful criterion to spot local systems. For a proof, we refer to [Ach20], Lemma 1.7.5.

Lemma 1.7. *Let $\mathcal{F}$ be a presheaf on X, let $U \subset X$ be open. Consider the following two conditions:*

1. *Every $x \in U$ has a basis of neighbourhoods $V \subset U$ such that $\mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism.*

2. *The sheaf $\mathcal{F}^+|_U$ is isomorphic to the constant sheaf with value $\mathcal{F}(U)$.*

Condition (1) implies condition (2). If $\mathcal{F}$ is a sheaf and U is connected and locally connected, the converse also holds.

2 The monodromy representation

We will now investigate how our concept of local systems ties up with representations of the fundamental group of spaces. Since paths and homotopies are all about unit intervals and cubes, the following will be helpful:

Lemma 2.1. *Every local system on $[0, 1]$ or $[0, 1] \times [0, 1]$ is a constant sheaf.*

Proof. We only give the proof for $[0, 1]$. Let $\mathcal{L}$ be a local system on $[0, 1]$. Because of compactness, we can find finitely many intervals $U_1, ..., U_n$ such that $\mathcal{L}|_{U_i}$ are constant sheaves. Withous loss of generality assume that $0 \in U_1$ and that U_i contains the right endpoint of $\overline{U_{i-1}}$. Define $V_i := U_1 \cup \cdots \cup U_i$. Hence the V_i are connected open sets with $0 \in V_i$, and obviously $V_n = [0, 1]$. By induction, we will show that $\mathcal{L}|_{V_i}$ are all constant sheaves.

The induction beginning holds by assumption. Now let $i > 1$. V_{i-1} and U_i are both intervals, hence $V_{i-1} \cap U_i$ is also connected (and locally connected). By Remark 1.3, the restriction maps

$$\mathcal{L}(V_{i-1}) \longrightarrow \mathcal{L}(V_{i-1} \cap U_i) \longleftarrow \mathcal{L}(U_i)$$

are both isomorphisms. From $V_i = V_{i-1} \cup U_i$ we see that $\mathcal{L}(V_i) \xrightarrow{\cong} \mathcal{L}(V_{i-1})$ and $\mathcal{L}(V_i) \xrightarrow{\cong} \mathcal{L}(U_i)$ are isomorphisms. By Lemma 1.7, $\mathcal{L}|_{V_i}$ is constant. □

Given a local system $\mathcal{L}$ on X and a path $\gamma : [0,1] \to X$, we define the map $\rho(\gamma) : \mathcal{L}_{\gamma(1)} \to \mathcal{L}_{\gamma(0)}$ via the composition

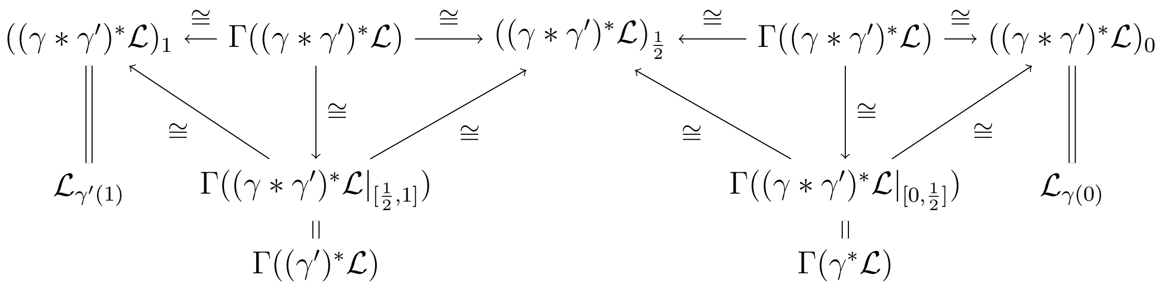
$$\mathcal{L}_{\gamma(1)} \cong (\gamma^*\mathcal{L})_1 \xleftarrow{\cong} \Gamma(\gamma^*\mathcal{L}) \xrightarrow{\cong} (\gamma^*\mathcal{L})_0 \cong \mathcal{L}_{\gamma(0)}$$

where the middle two isomorphisms come from Remark 1.3, since $\gamma^*\mathcal{L}$ is a local system on $[0,1]$, hence constant by the above lemma. To turn this idea into a representation of the fundamental group, this better behave well under concatenation and homotopies of paths, and indeed it does.

Lemma 2.2. *Let $\mathcal{L}$ be a local system on X. The followings holds:*

1. *If $\gamma : [0,1] \to X$ is a constant path in x, then $\rho(\gamma) = \mathrm{id}_{\mathcal{L}_x}$.*
2. *If $\gamma, \gamma' : [0,1] \to X$ are paths with $\gamma(1) = \gamma'(0)$, then $\rho(\gamma * \gamma') = \rho(\gamma) \circ \rho(\gamma')$.*
3. *If $\gamma, \gamma' : [0,1] \to X$ are homotopic paths with common endpoints, then $\rho(\gamma) = \rho(\gamma')$.*

Proof. Part 1 is obvious. For part 2, one can consider the following commutative diagram:



Now suppose H is a homotopy between γ and γ', explicitly $H : [0,1] \times [0,1] \to X$ such that $H(t,0) = \gamma(t)$, $H(t,1) = \gamma'(t)$, $H(0,s) = x_0$ and $H(1,s) = x_1$, where $x_0 = \gamma(0) = \gamma'(0)$ and $x_1 = \gamma(1) = \gamma'(1)$. Then we get the following commutative diagram, from with part 3 follows.

$$\begin{array}{ccccc}
\mathcal{L}_{x_1} & \xleftarrow{\cong} & \Gamma(\gamma^*\mathcal{L}) = \Gamma((H^*\mathcal{L})|_{[0,1]\times\{0\}}) & \xrightarrow{\cong} & \mathcal{L}_{x_0} \\
\| & & \uparrow \wr & & \| \\
\Gamma((H^*\mathcal{L})|_{\{1\}\times[0,1]}) & \xleftarrow{\cong} & \Gamma(H^*\mathcal{L}) & \xrightarrow{\cong} & \Gamma((H^*\mathcal{L})|_{\{0\}\times[0,1]}) \\
\| & & \downarrow \wr & & \| \\
\mathcal{L}_{x_1} & \xleftarrow{\cong} & \Gamma((\gamma')^*\mathcal{L}) = \Gamma((H^*\mathcal{L})|_{[0,1]\times\{1\}}) & \xrightarrow{\cong} & \mathcal{L}_{x_0}
\end{array}$$

□

Definition 2.3. Let $\mathcal{L}$ be a local system on a space X. For any basepoint $x_0 \in X$, the above construction defines the so called monodromy representation of $\pi_1(X, x_0)$ on the stalk $\mathcal{L}_{x_0}$, given by $[\gamma] \mapsto \rho(\gamma)$.

We get the functor

$$\mathrm{Mon}_{x_0} : \mathrm{Loc}\, X \to \mathbb{C}[\pi_1(X, x_0)]\text{-Mod} \tag{1}$$

Obviously it sends finite-type local systems to finite-dimensional representations.

Theorem 2.4. *Let X be connected, locally path-connected and semilocally simply connected. Then 1 is an equivalence of categories, as is its restriction to finite-type local systems.*

Proof. Since connected and locally path-connected implies path-connected, we once and for all fix a path $\alpha_x : [0, 1] \to X$ such that $\alpha_x(0) = x_0$ and $\alpha_x(1) = x$. We choose α_{x_0} to be the constant path.

Given a $\mathbb{C}[\pi_1(X, x_0)]$-module M, we define for $U \subset X$ open:

$$Q(M)(U) := \{s : U \to M \mid \forall \gamma : [0, 1] \to U \ s(\gamma(0)) = [\alpha_{\gamma(0)} * \gamma * \alpha^{-1}_{\gamma(1)}] \cdot s(\gamma(1))\}$$

Then one has to check that this is a local system on X and is inverse to the functor Mon_{x_0}. □

For calculations, the following proposition is of much importance.

Proposition 2.5. *Let $f : X \to Y$ be a continuous map of spaces.*

1. *Suppose that f preserves basepoints $x_0 \in X$ and $y_0 \in Y$ and that X and Y are connected, locally path-connected and semilocally simply connected. Then we have a natural isomorphism for $\mathcal{L} \in \mathrm{Loc}\, Y$:*

$$\mathrm{Mon}_{x_0}(f^*\mathcal{L}) \cong \mathbf{Res}^{\mathbb{C}[\pi_1(Y,y_0)]\text{-Mod}}_{\mathbb{C}[\pi_1(X,x_0)]\text{-Mod}} \mathrm{Mon}_{y_0} \mathcal{L}$$

2. *If f is a covering map and X and Y are locally path-connected and locally simply connected. Then $^\circ f_*$ restricts to an exact functor $f_* : \mathrm{Loc}\, X \to \mathrm{Loc}\, Y$.*

3. *Suppose that f is a covering map preserving basepoints $x_0 \in X$ and $y_0 \in Y$, and that X and Y are connected, locally path-connected and locally simply connected. Then there is a natural isomorphism for $\mathcal{L} \in \mathrm{Loc}\, X$:*

$$\mathrm{Mon}_{y_0}(f_*\mathcal{L}) \cong \mathrm{Hom}_{\mathbb{C}[\pi_1(X,x_0)]}(\mathbb{C}[\pi_1(Y, y_0)], \mathrm{Mon}_{x_0}(\mathcal{L}))$$

Proof. For part 1 and 3, note that we get an algebra map $\mathbb{C}[\pi_1(X, x_0)] \to \mathbb{C}[\pi_1(Y, y_0)]$ induced by $[\gamma] \mapsto [f \circ \gamma]$. Part 1 then follows from the observation that the action of $[f \circ \gamma]$ on $\mathcal{L}_{y_0} \cong (f^*\mathcal{L})_{x_0}$ is the same as the action of $[\gamma]$ on $f^*\mathcal{L}$ via the isomorphism $(f \circ \gamma)^*\mathcal{L} \cong \gamma^*(f^*\mathcal{L})$. Given part 2, part 3 is a formal consequence from Theorem

2.4 and that f_* is right adjoint to f^*, whereas $\mathrm{Hom}_{\mathbb{C}[\pi_1(X,x_0)]}(\mathbb{C}[\pi_1(Y,y_0)],-)$ is right adjoint to $\mathbf{Res}^{\mathbb{C}[\pi_1(Y,y_0)]\text{-Mod}}_{\mathbb{C}[\pi_1(X,x_0)]\text{-Mod}}$.

As for part 2, we first show that ${}^\circ f_*$ sends local systems to local systems. Let $\mathcal{L}$ be a local system on X. Pick $y \in Y$ and $U \subset Y$ an open neighbourhood such that $f^{-1}(U)$ is the disjoint union of open sets $(V_\alpha)_\alpha$ all of which are homeomorphic to U. By shrinking U if necessary, we may assume U and hence all V_α to be connected and simply connected. Due to the construction of the inverse functor in Theorem 2.4, any local system on U or on V_α is constant.

Define $M_\alpha := \mathcal{L}(V_\alpha)$, so that $\mathcal{L}|_{V_\alpha} \cong \underline{M_\alpha}_{V_\alpha}$. Then we find

$$ {}^\circ f_*(\mathcal{L})(U) = \mathcal{L}(f^{-1}(U)) = \prod_\alpha M_\alpha, $$

and this formula still holds if we restrict to some smaller connected $U' \subset U$. Lemma 1.7 implies that ${}^\circ f_*\mathcal{L}$ is a local system.

To proof exactness, it suffices to check that ${}^\circ f_*$ sends surjective maps to surjective maps. Let $\phi : \mathcal{L} \to \mathcal{M}$ be a surjective map between local systems on X. Because of the above and Proposition 1.5, $\phi|_{V_\alpha}$ is surjective. Then $({}^\circ f_*\phi)_U$ can be identified with $\prod_\alpha \phi|_{V_\alpha}$, which is surjective. □

Example 2.6. As a computational example, we consider $X = \mathbb{C} \setminus \{0\}$ and the map $f : X \to X$, $z \mapsto z^2$. This clearly satifies the assumptions of Proposition 2.5, part 3. We choose 1 as a base point and consider the pushforward of the constant sheaf $\underline{\mathbb{C}}_X$.

Then we have $\mathbb{C}[\pi_1(X,1)] \cong \mathbb{C}[t,t^{-1}]$ with the endomorphism of $\mathbb{C}$-algebras given by $t \mapsto t^2$. Proposition 2.5 yields

$$ \mathrm{Mon}_1(f_*\underline{\mathbb{C}}_X) \cong \mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(\mathbb{C}[t,t^{-1}],\mathbb{C}), $$

where t acts by identity on $\mathbb{C}$ and on $\mathbb{C}[t,t^{-1}]$ by multiplication with t^2. Now $\mathrm{Mon}_1(f_*\underline{\mathbb{C}}_X)$ turns into a $\mathbb{C}[t,t^{-1}]$-module via the rule $(t\phi)(x) = \phi(tx)$ for any $x \in \mathbb{C}[t,t^{-1}]$. Note that such a ϕ is uniquely determined by $\phi(1)$ and $\phi(t)$, and that $t\phi(t) = \phi(t^2) = \phi(1)$ due to $\mathbb{C}[t,t^{-1}]$-linearity.

Hence $\mathrm{Mon}_1(f_*\underline{\mathbb{C}}_X)$ is a two-dimensional representation over $\mathbb{C}$ with basis (e_1, e_2) such that t acts by swapping the basis vectors.

3 Proper pushforward

We say a continous map $f : X \to Y$ is proper if it is universally closed. If both spaces are locally compact (so in particular Hausdorff), this is equivalent to preimages of compact sets being compact, or even to all fibers of f being compact. This will be our main setting in the future.

Definition 3.1. Let $f : X \to Y$ be continuous and let $\mathcal{F}$ be a sheaf on X. The proper push-forward of $\mathcal{F}$ along f is the sheaf ${}^\circ f_!\mathcal{F} \in \mathrm{Sh}\, Y$ defined by

$$ {}^\circ f_!\mathcal{F}(U) := \{s \in \mathcal{F}(f^{-1}(U) \mid f|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to U \text{ is proper}\} $$

Proper push-forward is a functor $\mathrm{Sh}\, X \to \mathrm{Sh}\, Y$. Given a morphism $\alpha : \mathcal{F} \to \mathcal{G}$, we first get a morphism ${}^\circ f_* \alpha : {}^\circ f_* \mathcal{F} \to {}^\circ f_* \mathcal{G}$, given on U by $\alpha_{f^{-1}(U)}$. If we now pick some $s \in \mathcal{F}(f^{-1}(U))$ such that $f|_{\mathrm{supp}\, s}$ is proper, then

$$\mathrm{supp}\, \alpha_{f^{-1}(U)}(s) = \{x \in f^{-1}(U) \mid \alpha_x(s_x) \neq 0\} \subset \mathrm{supp}\, s$$

and this inclusion of a closed subset is proper.

Definition 3.2. Let $\mathcal{F}$ be a sheaf on a space X. We define its global sections with compact support as

$$\Gamma_c(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) : \mathrm{supp}\, s \text{ is compact}\}$$

Remark 3.3. As we have manifestly defined ${}^\circ f_! \mathcal{F}$ as a subfunctor of ${}^\circ f_* \mathcal{F}$, we get a natural transformation

$$ {}^\circ f_! \mathcal{F} \to {}^\circ f_* \mathcal{F} \tag{2}$$

Our next goal is to describe the stalks of ${}^\circ f_! \mathcal{F}$.

Lemma 3.4. *Let X be a space, Z a subspace and $\mathcal{F}$ a sheaf on X. Consider the canonical morphism*

$$\psi : \varinjlim \Gamma(U, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|_Z)$$

where U ranges over all open neighbourhoods of Z. ψ is injective, and if X is Hausdorff and Z is compact, then ψ is an isomorphism.

Proof. Suppose $s \in \Gamma(U, \mathcal{F})$ is zero in $\Gamma(Z, \mathcal{F}|_Z)$, then $s_x = 0$ for all $x \in Z$. Hence s must already be zero on an open neighbourhood of Z.

Now let $s \in \Gamma(Z, \mathcal{F}|_Z)$ and assume Z is compact. Hence we can find a finite open cover $\bigcup_{i=1}^n U_i \supset Z$ and $s_i \in \Gamma(U_i, \mathcal{F})$ such that $s|_{U_i \cap Z} = s_i|_{U_i \cap Z}$. We can further find open V_i such that $\bigcup_{i=1}^n V_i \supset Z$ and $\overline{V_i} \subset U_i$. For $x \in X$, let $I(x) = \{i \in \{1, ..., n\} : x \in \overline{V_i}\}$ and

$$W = \{x \in \bigcup_{i=1}^n V_i : s_{ix} = s_{jx} \text{ for any } i, j \in I(x)\}$$

Every x has a neighbourhood W_x such that $I(y) \subset I(x)$ for every $y \in W_x$, hence W is open. By construction W contains Z. Because of $s_i|_{W \cap V_i \cap V_j} = s_j|_{W \cap V_i \cap V_j}$, we can glue them to a section $\tilde{s} \in \Gamma(W, \mathcal{F})$. This section satisfies $\tilde{s}|_{W \cap V_i} = s_i|_{W \cap V_i}$, and hence $\psi(\tilde{s}) = s$. □

Proposition 3.5. *Suppose X and Y are locally compact and $f : X \to Y$ is continuous, and let $\mathcal{F} \in \mathrm{Sh}\, X$. Then for all $y \in Y$, the canonical morphism*

$$\alpha : ({}^\circ f_! \mathcal{F})_y \to \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$$

is an isomorphism.

Proof. For injectivity, let V be an open neighbourhood of y and $t \in \Gamma(V, {}^\circ f_! \mathcal{F})$, which is defined by some $s \in \Gamma(f^{-1}(V), \mathcal{F})$. If $\alpha(t) = 0$, then $\operatorname{supp} s \cap f^{-1}(y) \neq \emptyset$, hence $y \notin f(\operatorname{supp} s)$, which is closed. Pick some neighbourhood V' of y inside V such that $f(\operatorname{supp} s) \cap V' = \emptyset$, hence $\operatorname{supp} s \cap f^{-1}(V') = \emptyset$, hence $t = 0$.

For surjectivity, let $s \in \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$ and $K = \operatorname{supp} s$, which is compact. By Lemma 3.4 we find an open neighbourhood U of K and $t \in \Gamma(U, \mathcal{F})$ such that $t|_K = s|_K$. By shrinking U, we can further assume $t|_{U \cap f^{-1}(y)} = s|_{U \cap f^{-1}(y)}$. Let V be an open neighbourhood of K such that $\overline{V} \subset U$ is compact. Then $y \notin f(\overline{V} \cap \operatorname{supp} t \setminus V)$, which is compact, hence we find an open neighbourhood W of y with $W \cap f(\overline{V} \cap \operatorname{supp} t \setminus V) = \emptyset$, which implies $f^{-1}(W) \cap \overline{V} \cap \operatorname{supp} t \subset V$.

Now we define $\tilde{s} \in \Gamma(f^{-1}(W), \mathcal{F})$ by

$$\begin{aligned} \tilde{s}|_{f^{-1}(W) \setminus (\operatorname{supp} t \cap \overline{V})} &= 0 \\ \tilde{s}|_{f^{-1}(W) \cap V} &= t|_{f^{-1}(W) \cap V} \end{aligned}$$

This is indeed well-defined, and one can check that f is proper on $\operatorname{supp} \tilde{s}$ and that $\alpha(\tilde{s}) = s$. □

The description of the stalks of ${}^\circ f_! \mathcal{F}$ immediately shows that ${}^\circ f_!$ is a left-exact functor, since pullback is exact and the functor Γ_c is left-exact just as Γ is.

Definition 3.6. Let $f : X \to Y$ be continuous with X and Y locally compact. By $f_!$ we denote the derived functor of ${}^\circ f_!$:

$$f_! = R({}^\circ f_!) : D^+(X) \to D^+(Y)$$

Definition 3.7. Let X be locally compact. A sheaf $\mathcal{F}$ on X is called c-soft if for every compact $K \subset X$, the map $\Gamma(X, \mathcal{F}) \to \Gamma(K, \mathcal{F}|_K)$ is surjective.

Why do we care about c-soft sheaves? If $f : X \to Y$ is continuous with X and Y locally compact, then c-soft sheaves are an adapted class for ${}^\circ f_!$, and ${}^\circ f_!$ takes c-soft sheaves on X to c-soft sheaves on Y. For a proof we refer to [KS94], 2.5.6 to 2.5.9.

Proposition 3.8. *Suppose X, Y and Z are locally compact and $f : X \to Y$, $g : Y \to Z$ are continuous. Then for $\mathcal{F} \in D^+(X)$, there is a natural isomorphism*

$$g_! f_! \mathcal{F} \cong (g \circ f)_! \mathcal{F}$$

Proof. For the abelian version, we unravel the definitions:

$${}^\circ(g \circ f)_! \mathcal{F}(U) = \{ s \in \Gamma(f^{-1}(g^{-1}(U)), \mathcal{F}) \mid g \circ f : \operatorname{supp} s \to U \text{ proper} \} \quad (3)$$

$${}^\circ g_! {}^\circ f_! \mathcal{F}(U) = \{ t \in \Gamma(g^{-1}(U), {}^\circ f_! \mathcal{F}) \mid g : \operatorname{supp} t \to U \text{ proper} \}, \quad (4)$$

where t is itself given by some $s \in \Gamma(f^{-1}(g^{-1}(U)), \mathcal{F})$ such that $f : \operatorname{supp} s \to g^{-1}(U)$ is proper.

Now suppose we have a section t in 4 given by s as above. Then because $f : \operatorname{supp} s \to U$ is proper, $f(\operatorname{supp} s) = \operatorname{supp} t$ and hence $g \circ f : \operatorname{supp} s \to U$ is also proper.

On the other hand, suppose we have a section u in 3 given by $s \in \Gamma(f^{-1}(g^{-1}(U)), \mathcal{F})$ such that $g \circ f : \operatorname{supp} s \to U$ is proper. This also defines a section $t \in \Gamma(g^{-1}(U), {}^{\circ}f_*\mathcal{F})$.

We first show that $f : \operatorname{supp} s \to g^{-1}(U)$ is proper. For that, pick $y \in g^{-1}(U)$, then by assumption $f^{-1}(g^{-1}(g(y)))$ is compact and contains $f^{-1}(y)$, which is closed and hence also compact. This shows that $t \in \Gamma(g^{-1}(U), {}^{\circ}f_!\mathcal{F})$ and also that $f(\operatorname{supp} s) = \operatorname{supp} t$.

Now we show that $g : \operatorname{supp} t \to U$ is proper. For that, we pick $y \in U$, then by the above $g^{-1}(y) \cap \operatorname{supp} t = g^{-1}(y) \cap f(\operatorname{supp} s)$. This shows that $g^{-1}(y) \cap \operatorname{supp} t \subset f(f^{-1}(g^{-1}(y) \cap \operatorname{supp} t))$, and the latter is compact, hence $g^{-1}(y) \cap \operatorname{supp} t$ is also compact.

All of this shows that u is also a section in 4.

For the derived version, we make use of the fact that c-soft sheaves are an adapted class for ${}^{\circ}f_!$ and that it sends c-soft sheaves to c-soft sheaves. The result follows by general facts about derived functors. □

Proposition 3.9. *Suppose X and Y are locally compact and $f : X \to Y$ is continuous, and let $\mathcal{F} \in D^+(X)$. Then for all $y \in Y$, we have*

$$(f_!\mathcal{F})_y \cong R\Gamma_c(\mathcal{F}|_{f^{-1}(y)})$$

Proof. The abelian-category version was shown in Proposition 3.5. For the derived version, we replace $\mathcal{F} \in D^+(X)$ by a c-soft resolution. □

Theorem 3.10 (Projection formula)**.** *Let $f : X \to Y$ be a continuous map of locally compact spaces and let $\mathcal{F} \in D^+(X)$ and $\mathcal{G} \in D^+(Y)$, then we have a natural isomorphism*

$$f_!\mathcal{F} \otimes \mathcal{G} \xrightarrow{\cong} f_!(\mathcal{F} \otimes f^*\mathcal{G})$$

Proof. See [Ach20], Theorem 1.4.9. □

4 Proper base change

For the whole section, we are given a Cartesian diagram of topological spaces of the form

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array} \tag{5}$$

with $h := f \circ g' = g \circ f'$.

Lemma 4.1. *In the Cartesian square 5, for $\mathcal{F} \in D^+(X')$ there is a natural transformation*

$$\begin{array}{ccccc} f_! g'_* \mathcal{F} & \longrightarrow & & \longrightarrow & g_* f'_! \mathcal{F} \\ \downarrow 2 & & & & \downarrow 2 \\ f_* g'_* \mathcal{F} & \xrightarrow{\cong} & h_* \mathcal{F} & \xrightarrow{\cong} & g_* f'_* \mathcal{F} \end{array}$$

Proof. For the abelian-category version, we have to show that there is a (necessarily unique) natural transformation of the form

$$\begin{array}{ccc} {}^\circ f_! {}^\circ g'_* \mathcal{F} & \dashrightarrow & {}^\circ g_* {}^\circ f'_! \mathcal{F} \\ & \searrow \quad \swarrow & \\ & {}^\circ h_* \mathcal{F} & \end{array}$$

Pick $s \in ({}^\circ f_! {}^\circ g'_* \mathcal{F})(U)$ which is defined by some $v \in \mathcal{F}(h^{-1}(U))$, which in turn defines a section $u \in ({}^\circ g'_* \mathcal{F})(f^{-1}(U))$ such that $f : \operatorname{supp} u \to U$ is proper. Consider the base change

$$\begin{array}{ccc} W & \xrightarrow{g'} & \operatorname{supp} u \\ f' \downarrow & & \downarrow f \\ g^{-1}(U) & \xrightarrow[g]{} & U \end{array}$$

where W is the appropriate subspace in X'. Since properness is preserved under base change, the map $W \to g^{-1}(U)$ is proper. However, $\operatorname{supp} v \subset W$ is a closed subset, hence $f' : \operatorname{supp} v \to g^{-1}(U)$ is also proper. Hence s is also a section of ${}^\circ g_* {}^\circ f'_! \mathcal{F}$.

Passing to derived functors, we get the following diagram as we in general only get a natural transformation $R(F \circ G) \to RF \circ RG$:

$$\begin{array}{ccccccc} f_! g'_* \mathcal{F} & \xleftarrow{m_1} & R({}^\circ f_! {}^\circ g'_*)(\mathcal{F}) & \longrightarrow & R({}^\circ g_* {}^\circ f'_!)(\mathcal{F}) & \xrightarrow{m_2} & g_* f'_! \mathcal{F} \\ \downarrow 2 & & \downarrow 2 & & \downarrow 2 & & \downarrow 2 \\ f_* g'_* \mathcal{F} & \xleftarrow{m_3} & R({}^\circ f_* {}^\circ g'_*)(\mathcal{F}) & = & R({}^\circ g_* {}^\circ f'_*)(\mathcal{F}) & \xrightarrow{m_4} & g_* f'_* \mathcal{F} \end{array}$$

We already know the natural transformations m_3 and m_4 are isomorphisms. Since ${}^\circ g'_*$ takes injective sheaves to injective sheaves, m_1 is also an isomorphism. Thus we get the desired diagram - note that m_2 need not be an isomorphism. □

Lemma 4.2. *In the Cartesian square 5, for $\mathcal{F} \in D^+(X)$ there is a natural commutative diagram*

$$\begin{array}{ccc} g^* f_! \mathcal{F} & \longrightarrow & f'_!(g')^* \mathcal{F} \\ \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\ g^* f_* \mathcal{F} & \longrightarrow & f'_*(g')^* \mathcal{F} \end{array}$$

Proof. Apply Lemma 4.1 to $(g')^*\mathcal{F}$, apply g^* to the resulting diagram and use the adjunction maps $\mathrm{id} \to g'_*(g')^*$ and $g^* g_* \to \mathrm{id}$. We thus obtain the following commutative diagram

$$\begin{array}{ccccccc} g^* f_! \mathcal{F} & \longrightarrow & g^* f_! g'_*(g')^* \mathcal{F} & \longrightarrow & g^* g_* f'_!(g')^* \mathcal{F} & \longrightarrow & f'_!(g')^* \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ g^* f_* \mathcal{F} & \longrightarrow & g^* f_* g'_*(g')^* \mathcal{F} & \xrightarrow{\cong} & g^* g_* f'_*(g')^* \mathcal{F} & \longrightarrow & f'_*(g')^* \mathcal{F} \end{array}$$

□

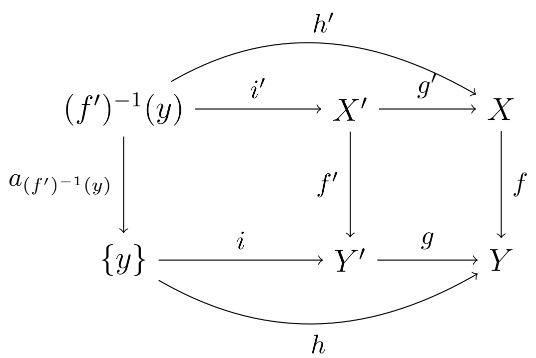
Theorem 4.3 (Proper base change)**.** *Consider the Cartesian square 5 and assume all spaces are locally compact. For any $\mathcal{F} \in D^+(X)$, the base change map $g^* f_! \mathcal{F} \to f'_!(g')^* \mathcal{F}$ from Lemma 4.2 is an isomorphism.*

Proof. We first note an important special case, namely when g is the inclusion of a point $\{y\}$ into Y and X' is identified with $f^{-1}(y) \subset X$. In this case the base change map is simply the identity

$$(f_! \mathcal{F})_y \cong R\Gamma_c(\mathcal{F}|_{f^{-1}(y)})$$

which we have already proven in Proposition 3.9.

In the general case, it is enough to that the induced map on stalks is an isomorphism for every $y \in Y'$. Consider the following Cartesian squares



This gives rise to a commutative diagram

$$\begin{array}{ccccc} (g^* f_! \mathcal{F})_y & \longrightarrow & (f'_!(g')^* \mathcal{F})_y & \xrightarrow{b_1} & R\Gamma_c(((g')^*\mathcal{F})|_{(f')^{-1}(y)}) \\ \downarrow \wr\| & & & & \downarrow \|\wr \\ h^* f_! \mathcal{F} & & \xrightarrow{\quad b_2 \quad} & & R\Gamma_c((h')^* \mathcal{F}) \end{array}$$

The maps b_1 and b_2 are isomorphisms by the discussed special case above, hence the map in question is also an isomorphism. □

Theorem 4.4 (Proper base change)**.** *In the Cartesian square 5 assume that f is proper. For any $\mathcal{F} \in D^+(X)$, the base change map $g^* f_* \mathcal{F} \to f'_*(g')^* \mathcal{F}$ from Lemma 4.2 is an isomorphism.*

Proof. The proof follows the same pattern as that of Theorem 4.3. One first proofs as a special case that

$$(f_* \mathcal{F})_y \cong R\Gamma(\mathcal{F}|_{f^{-1}(y)})$$

by proving the abelian-category version and then using a flabby resolution for $\mathcal{F}$. □

We can neither exchange $f_!$ with f_* in Theorem 4.3 nor drop the properness assumption in Theorem 4.4, as the following example shows.

Example 4.5. Consider the following Cartesian diagram.

$$\begin{array}{ccc} \emptyset & \xrightarrow{g'} & \mathbb{C} \setminus \{0\} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \{0\} & \xrightarrow[g]{} & \mathbb{C} \end{array}$$

where f and g are the obvious inclusions. Note that f is not proper. Consider the constant sheaf $\underline{\mathbb{C}}$ on $\mathbb{C} \setminus \{0\}$. Then $f'_*(g')^* \underline{\mathbb{C}} = 0$, but we saw in the last talk that $g^* f_* \mathcal{F} = (f_* \mathcal{F})_0 \cong R\Gamma(\underline{\mathbb{C}})$, which is not 0.

5 The right adjoint to $f_!$

In this section we will get to know the right adjoint to $f_!$. This is the first instance that we will see of a functor defined between derived categories that is itself not a derived functor.

Definition 5.1. Let X be locally compact. We say X has c-soft dimension $\leq n$ if every sheaf $\mathcal{F} \in \operatorname{Sh} X$ admits a c-soft resolution of length at most n.

Proposition 5.2. *Let M be a real n-dimensional manifold and $X \subset M$ locally closed. Then X has c-soft dimension $\leq n$.*

Remark 5.3. If X has c-soft dimension $\leq n$ and $f : X \to Y$ is continuous with Y locally compact, then $^\circ f_!$ has cohomological dimension $\leq n$. Hence in this setting it makes sense to speak of the derived functor $f_! : D^-(X) \to D^-(Y)$. We will need this in the following.

Theorem 5.4. *Let $f : X \to Y$ be continuous with X and Y locally compact, and assume $^\circ f_!$ has finite cohomological dimension. Then there exists a triangulated functor*

$$f^! : D^+(Y) \to D^+(X)$$

such that for $\mathcal{F} \in D^-(X)$ and $\mathcal{G} \in D^+(Y)$ the following holds:

$$R\mathcal{H}\mathrm{om}(f_!\mathcal{F}, \mathcal{G}) \cong f_* R\mathcal{H}\mathrm{om}(\mathcal{F}, f^!\mathcal{G})$$
$$R\,\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) \cong R\,\mathrm{Hom}(\mathcal{F}, f^!\mathcal{G})$$
$$\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f^!\mathcal{G})$$

Proof. First some heuristics to motivate the construction of $f^!$: Assume the theorem to hold and that for $\mathcal{G} \in \mathrm{Sh}\,Y$, $f^!\mathcal{G}$ is again a sheaf. To describe $f^!\mathcal{G}$, we have to describe its section for some open subset $U \subset X$. Write $j_U : U \to X$ for the inclusion. Then

$$(f^!\mathcal{G})(U) \cong \Gamma(j_U^! f^!\mathcal{G}) \cong \mathrm{Hom}(\underline{\mathbb{C}}_U, j_U^! f^!\mathcal{G}) \cong \mathrm{Hom}(f_! j_{U!} \underline{\mathbb{C}}_U, \mathcal{G}).$$

In more generality, if $\mathcal{F}$ is a sheaf on X, then $\mathcal{H}\mathrm{om}(\mathcal{F}, f^!\mathcal{G})$ is described by

$$\mathcal{H}\mathrm{om}(\mathcal{F}, f^!\mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|_U, (f^!\mathcal{G})|_U) \cong \mathrm{Hom}(j_{U!} j_U^* \mathcal{F}, f^!\mathcal{G}) \cong \mathrm{Hom}(f_!(\mathcal{F} \otimes j_{U!} \underline{\mathbb{C}}_U), \mathcal{G}),$$

which is a formula that doesn't make use of $f^!$.

As for the construction of $f^!$, we first fix a finite c-soft (and in our setting automatically flat) resolution $\mathcal{K}$ of the constant sheaf $\underline{\mathbb{C}}_X$.

For $\mathcal{F} \in Ch^-(\mathrm{Sh}\,X))$ and $\mathcal{G} \in Ch^+(\mathrm{Sh}\,Y)$ we let $\mathcal{E}(\mathcal{F}, \mathcal{G})$ be the following chain complex of presheaves on X

$$\mathcal{E}(\mathcal{F}, \mathcal{G})(U) = \mathrm{chHom}(^\circ f_!(\mathcal{F} \otimes j_{U!}(\mathcal{K}|_U)), \mathcal{G})$$

where $j_U : U \to X$ is the inclusion of $U \subset X$. Then one can show that this is a chain complex of sheaves, so that we get a functor

$$\mathcal{E} : K^-(\mathrm{Sh}\,X)^{\mathrm{op}} \times K^+(\mathrm{Sh}\,Y) \to K^+(\mathrm{Sh}\,X)$$

There are the following natural isomorphisms:

$$\mathcal{E}(\mathcal{F}, \mathcal{G}) \cong \mathrm{ch}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{E}(\underline{\mathbb{C}}_X, \mathcal{G}))$$

$$\mathrm{ch}\mathcal{H}\mathrm{om}({}^\circ f_!(\mathcal{F}\otimes\mathcal{K}),\mathcal{G}) \cong {}^\circ f_*\mathcal{E}(\mathcal{F},\mathcal{G})$$

Then one shows that this functor admits a derived functor

$$R\mathcal{E} : D^-(X)^{\mathrm{op}} \times D^+(Y) \to D^+(X)$$

For $\mathcal{F} \in D^-(X)$ and $\mathcal{G} \in D^+(Y)$ there are natural isomorphisms

$$R\mathcal{E}(\mathcal{F},\mathcal{G}) \cong R\mathcal{H}\mathrm{om}(\mathcal{F}, R\mathcal{E}(\underline{\mathbb{C}}_X,\mathcal{G}))$$

$$R\mathcal{H}\mathrm{om}(f_!\mathcal{F},\mathcal{G}) \cong f_*R\mathcal{E}(\mathcal{F},\mathcal{G})$$

We define the functor $f^! : D^+(Y) \to D^+(X)$ by setting

$$f^!\mathcal{G} := R\mathcal{E}(\underline{\mathbb{C}}_X,\mathcal{G}),$$

and by the above isomorphisms, this indeed gives us the right adjoint to $f_!$. $\square$

It needs to be emphasized that this is our first encounter with a triangulated functor between derived categories that isn't the derived functor of a functor between abelian categories. The following three statements about $f^!$ can all be deduced from Theorem 5.4 and from Proposition 3.8, respectively Theorem 4.3 respectively the projection formula.

Remark 5.5. Remember that for a continuous map $f : X \to Y$ of locally compact spaces, the projection formula says that for $\mathcal{F} \in D^+(X)$ and $\mathcal{G} \in D^+(Y)$, we have a natural isomorphism

$$f_!\mathcal{F} \otimes \mathcal{G} \xrightarrow{\cong} f_!(\mathcal{F} \otimes f^*\mathcal{G})$$

Proposition 5.6. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of locally compact spaces. Assume that ${}^\circ f_!$ and ${}^\circ g_!$ have finite cohomological dimension. Then for $\mathcal{F} \in D^+(Z)$, we have a natural isomorphism*

$$f^!g^!\mathcal{F} \cong (g \circ f)^!\mathcal{F}$$

Proposition 5.7. *Suppose we have a Cartesian square 5 between locally compact spaces. Assume ${}^\circ f_!$ has finite cohomological dimension.*

1. *For $\mathcal{F} \in D^+(Y)$, there is a natural map $(g')^*f^!\mathcal{F} \to (f')^!g^*\mathcal{F}$*

2. *For $\mathcal{F} \in D^+(Y')$ there is natural commutative diagram*

$$\begin{array}{ccc} g'_!(f')^!\mathcal{F} & \longrightarrow & f^!g_!\mathcal{F} \\ \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2} \\ g'_*(f')^!\mathcal{F} & \xrightarrow{\cong} & f^!g_*\mathcal{F} \end{array}$$

Proposition 5.8. *Let $f : X \to Y$ be a continuous map of locally compact spaces. Assume that ${}^\circ f_!$ has finite cohomological dimension. For $\mathcal{F} \in D^b(Y)$ and $\mathcal{G} \in D^+(Y)$ there is natural isomorphism*

$$f^! R\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \cong R\mathcal{H}\mathrm{om}(f^*\mathcal{F}, f^!\mathcal{G})$$

Proof. Note that we require $\mathcal{F} \in D^b(Y)$ because $\mathcal{F}$ has to be bounded below to be able to be put into $R\mathcal{H}$om, and bounded above for the projection formula to hold. □

References

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