My part of the talk will also be divided into two parts. ~~The first part concerns how~~
In the first part we will exhibit the proper pushforward functor in the case of a locally closed embedding and define its right adjoint — proper pullback, which in the setting of locally closed embedding is easier to define. We will learn how our functors interact ~~in~~ for open and closed embedding, resulting in the theorem about open-closed distinguished triangles.

The second part will be devoted to the derived sheaf Hom functor and to its left adjoint — derived tensor product. We will learn how these functors interact, in particular, we will learn the projection formula.
~~By the en~~
By the end of the talk, we will have 3 pairs of adjoint functors, and various results on how they interact together.

Open and closed embeddings

Fix an inclusion of a locally closed subset $h: Y \hookrightarrow X$.

<u>Lemma 1.3.1</u> Then ${}^{\circ}h_! F$ ($F \in Sh(Y,k)$) is naturally isomorphic to the sheafification of the presheaf ~~given by~~

$$\mathcal{U} \longmapsto \begin{cases} F(\mathcal{U} \cap Y) & \text{if } \mathcal{U} \cap \bar{Y} \subset Y \\ 0 & \text{else} \end{cases}$$

Moreover, ${}^{\circ}h_!$ is exact with stalks

$$({}^{\circ}h_! F)_x \cong \begin{cases} F_x, & x \in Y \\ 0 & \cancel{\text{else}}\ (x \notin Y) \end{cases}$$

locally closed implies that the stalks are such, which in turn implies exactness. because of the stalks it is called "extension by 0".

We obtain the functor $h_!: D^+(Y,k) \to D^+(X,k)$

Since ${}^{\circ}h_!$ is exact, we omit the circle "$\circ$" and write $h_!$

<u>Definition 1.3.2.</u> The functor of <u>restriction with supports</u> in Y is the functor

$${}^{\circ}h^!: Sh(X,k) \longrightarrow Sh(Y,k) \text{ given by}$$

$${}^{\circ}h^!(F)(\mathcal{U}) = \operatorname*{colim}_{\substack{V \subset X \text{ open} \\ V \cap \bar{Y} = \mathcal{U}}} \{ s \in F(V) \mid \operatorname{supp} s \subset \mathcal{U} \}$$

it defines a sheaf and not just a presheaf.

one remarkable property of $h_!$ is that for locally closed embeddings it has a rather down-to-earth right adjoint. (In general it is more difficult)

<u>Lemma 1.3.4</u> ${}^{\circ}h^!$ is left exact. exercise

Then its right derived functor $h^!: D^+(X,k) \to D^+(Y,k)$ exists.

<u>Lemma 1.3.3</u> If h is an open embedding, then $h^! = h^*$. follows immediately from Def. 1.3.2.

<u>Lemma ?.</u> If h is a closed embedding, then $h_! = h_*$.

because closed embeddings are proper, [illegible] so proper push coincides with usual push.

Theorem 1.3.6. For $\mathcal{F} \in D^+(Y,k)$, $\mathcal{G} \in D^+(X,k)$, there is a natural isomorphism

$$\mathrm{Hom}_{D^+(X,k)}(h_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{D^+(Y,k)}(\mathcal{F}, h^!\mathcal{G})$$

this is our second adjunction! the proof follows the pattern of the proof of ~~theorem~~ the first adjunction. As a remark, $h^!$ sends injectives to injectives.

Proposition 1.3.7. Let $k: W \hookrightarrow Y$ be another locally closed embedding.

1) $h_! \circ k_! \cong (h \circ k)_!$ ← by exactness computed object-wise
2) $h^! \circ k^! \cong (h \circ k)^!$ ← by uniqueness of right adjoints

Proposition 1.3.9 For any $\mathcal{F} \in D^+(Y,k)$, the natural maps

$$\mathcal{F} \to h^! h_! \mathcal{F} \to h^* h_! \mathcal{F} \quad \text{and} \quad h^! h_* \mathcal{F} \to h^* h_* \mathcal{F} \to \mathcal{F}$$

are all isomorphisms.

unit/counit is an iso
$\Rightarrow$ some functor is fully-faithful.

Corollary 1.3.11. Let $i: Z \hookrightarrow X$ be a closed embedding. ~~The~~ The functor $i_*: D^+(Z,k) \to D^+(X,k)$ is fully-faithful with essential image

$$\{\mathcal{F} \in D^+(X,k) \mid \mathrm{supp}\,\mathcal{F} \subset Z\}.$$

essential image left as an exercise

Theorem 1.3.10 Let $i: Z \to X$ be a closed embedding, let $j: U \to X$ be the complementary open embedding.

← the following theorem is the most important theorem in this part of ~~the~~ my talk.

1) $i^* \circ j_! = 0$, $i^! \circ j_* = 0$, $j^* \circ i_* = 0$.
2) For any $\mathcal{F} \in D^+(X,k)$, there is a ~~natural~~ natural distinguished triangle

$$j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to \qquad (1.3.4).$$

Moreover, if $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ is a distinguished triangle with $i^*\mathcal{F}' = 0$, $j^*\mathcal{F}''$, then it is canonically isomorphic to (1.3.4).

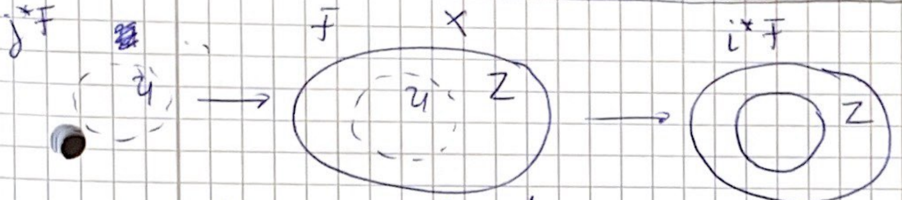
we continue on the next blackboard

3) For any $\mathcal F \in D^b(X,k)$, there is a natural distinguished triangle

(1.3.5) $\quad i_! i^! \mathcal F \to \mathcal F \to j_* j^* \mathcal F \to$

Moreover, if $\mathcal F' \to \mathcal F \to \mathcal F'' \to$ is a distinguished triangle with $i^! \mathcal F'' = 0$ and $j^* \mathcal F' = 0$, then it is canonically isomorphic to (1.3.5).

$\boxed{\text{In each triangle, all maps are adjunction maps.}}$



That is why open-closed.

Comment: for the proof of 1): ___ follows from stalks and exactness. ⇝ $i^! \circ j_*$ is right adjoint to $j^* \circ i_! = j^* i_*$, so it vanishes, too.

2) arises from the corresponding short exact sequence (check on stalks).

Task: compute $i^! \underline{k}_{\mathbb C}$, where $i : \circ \to \mathbb C$ is the inclusion of $\{0\}$. Consider $\circ \xrightarrow{i} \mathbb C \xrightarrow{a_{\mathbb C}} \circ$.

Consider the distinguished triangle

$$i_! i^! \underline k_{\mathbb C} \to \underline k_{\mathbb C} \to j_* j^* \underline k_{\mathbb C} \to$$

$a_{\mathbb C *}$ is triangulated

$$a_{\mathbb C *} i_! i^! \underline k_{\mathbb C} \to a_{\mathbb C *} \underline k_{\mathbb C} \to a_{\mathbb C *} j_* j^* \underline k_{\mathbb C} \to$$

$\quad \| \qquad\qquad \| \qquad\qquad \|$

needed! $\qquad a_{\mathbb C *} \underline k_{\mathbb C} \qquad a_{\mathbb C^\times *} \underline k_{\mathbb C^\times}$

les in cohomology

$$H^0(i^! \underline k_{\mathbb C}) \to k \to k \to$$
$$\to H^1(i^! \underline k_{\mathbb C}) \to 0 \to k \to H^2(i^! \underline k_{\mathbb C}) \to 0 \dots$$

Here is an application of open-closed triangles.

The cohomology is concentrated in degree 2! (applied singular cohomology of $\mathbb C$ and $\mathbb C^\times$).

open-closed triangles are probably the most powerful and basic computational tools for working with constructible sheaves.

The idea is to "separate" the stalks of $\mathcal F$ into an open part $\mathcal U$ and closed part Z.

← the "moreover" part follows from formal properties of distinguished triangles: it essentially says that any other such open-closed decomposition is the same.

3) is more complicated, but also arises from ses when $\mathcal F$ is flabby.

We now come to the second part of my (part) of the talk, which concerns the derived sheaf Hom and its left adjoint derived tensor product.

By the end of the talk, we will have 3 pairs of adjoint functors and various results on how they interact together.

Tensor product and sheaf Hom

Definition 1.4.1 For $\mathcal{F}, \mathcal{G} \in Sh(X,k)$, their <u>tensor product</u> $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of the presheaf $\mathcal{F} \otimes_{pre} \mathcal{G}$ given by

$$\mathcal{F} \otimes_{pre} \mathcal{G}(U) = \mathcal{F}(U) \otimes \mathcal{G}(U)$$

Their <u>sheaf Hom</u> is the sheaf $\mathcal{H}om(\mathcal{F},\mathcal{G})$

$$\mathcal{H}om(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}_{Sh(U,k)}(\mathcal{F}|_U, \mathcal{G}|_U)$$

Both functors generalize to complexes in the usual way. ~~we omit~~

Lemma 1.4.2 The functor $\otimes : Sh(X,k) \times Sh(X,k) \to Sh(X,k)$ is right exact in both variables, while $\mathcal{H}om : Sh(X,k)^{op} \times Sh(X,k) \to Sh(X,k)$ is left exact in both variables.

since tensoring commutes with taking filtered colimits, the stalks of the tensor is the tensor of stalks, and the right exactness follows from the stalks.

the left exactness of $\mathcal{H}om$ follows readily from definitions.

Since $Sh(X,k)$ has enough injectives, we have

$$R\mathcal{H}om : D^-(X,k)^{op} \times D^+(X,k) \to D^+(X,k)$$

Definition 1.4.3 $\mathcal{F} \in Sh(X,k)$ is <u>flat</u> if $\mathcal{F} \otimes (-) : Sh(X,k) \to Sh(X,k)$ is exact.

since we are working over $k = \mathbb{C}$, tensor is exact in both variables.

Using flat resolutions, we have

$$\overset{L}{\otimes} : D^-(X,k) \times D^-(X,k) \longrightarrow D^-(X,k)$$

If k has finite global dimension (e.g. $k = \mathbb{C}$), then we also have

$$\overset{L}{\otimes} : D^+(X,k) \times D^+(X,k) \longrightarrow D^+(X,k)$$

But $Sh(X,k)$ does not have enough projectives. ~~But~~ flat is left adapted to $\mathcal{F} \otimes (-)$.

Knowing how our functors interact with various adapted classes, we can prove a large number of natural isomorphisms among various compositions of derived (co)sheaf functors, by using Remark 1.2.6.

Proposition 1.4.4. 1) For $\mathcal{F} \in D^{-}(X,k)$, there is a natural isomorphism $\underline{k}_X \otimes \mathcal{F} \xrightarrow{\cong} \mathcal{F}$. If k has finite global dimension, same is true for $D^{+}(X,k)$.

2) For $\mathcal{F} \in D^{+}(X,k)$ there is natural isomorphism
$$R\mathcal{H}om(\underline{k}_X, \mathcal{F}) \cong \mathcal{F}$$

3) For $\mathcal{F} \in D(X,k)$ and $\mathcal{G} \in D^{+}(X,k)$ and for any open $U \subset X$, there is a natural isomorphism
$$R\mathcal{H}om(\mathcal{F},\mathcal{G})|_U \cong R\mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$$

The abelian-category versions of the isomorphisms are easy, and to prove the derived category versions one has to exhibit a suitable adapted class. We will give an example later.

Proposition 1.4.5 Let $f: X \to Y$ be a continuous map. For $\mathcal{F},\mathcal{G} \in D^{-}(Y,k)$, there is a natural isomorphism
$$f^*(\mathcal{F} \overset{L}{\otimes} \mathcal{G}) \cong f^*\mathcal{F} \overset{L}{\otimes} f^*\mathcal{G}.$$
If k has finite global dimension, same holds for $\mathcal{F},\mathcal{G} \in D^{+}(Y,k)$.

Proof sketch: since tensoring commutes with taking filtered colimits, the abelian category version holds true. Since $f^*(\text{flat}) = \text{flat}$, both sides can be computed object-wise by taking a flat resolution of $\mathcal{F}$. $\square$

Table 1.4.1

Example. $\underline{k''}_X \overset{L}{\otimes} \underline{k'}_X = a_X^* \underline{k''}_{pt} \overset{L}{\otimes} a_X^* \underline{k'}_{pt} \cong$
$\cong a_X^*(\underline{k'}_{pt} \overset{L}{\otimes} \underline{k'}_{pt}) =$
$= \underline{k' \otimes k'}_X .$

(M and N instead of k' and k'')

Proposition 1.4.6 For $\mathcal{F} \in D^{-}(X,k)$, $\mathcal{G} \in D^{+}(X,k)$, there is a natural isomorphism
$$R\Gamma(R\mathcal{H}om(\mathcal{F},\mathcal{G})) \xrightarrow{\cong} R\mathrm{Hom}(\mathcal{F},\mathcal{G})$$

The abelian-category version is clear, and for the derived version one can take an injective resolution of $\mathcal{G}$.

Proposition 1.4.7. Let $f: X \to Y$ be a continuous map. For $\mathcal{F} \in D(Y,k)$ and $G \in D^+(X,k)$ there are natural isomorphisms

$$f_* R\mathcal{H}om(f^*\mathcal{F}, G) \xrightarrow{\cong} R\mathcal{H}om(\mathcal{F}, f_*G),$$
$$R\mathrm{Hom}(f^*\mathcal{F}, G) \xrightarrow{\cong} R\mathrm{Hom}(\mathcal{F}, f_*G),$$
$$\mathrm{Hom}(f^*\mathcal{F}, G) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{F}, f_*G).$$

we have an upgrade of one of our previous theorems (Theorem 1.2.4).

the top iso holds on the level of ab categories (from definitions) inj res of G.

other isos follow from the top one subsequently by taking $R\Gamma$ and then H^0.

Theorem 1.4.8. For $\mathcal{F}, G \in D^-(X,k)$, $H \in D^+(X,k)$, there are natural isomorphisms

$$R\mathcal{H}om(\mathcal{F} \overset{L}{\otimes} G, H) \xrightarrow{\cong} R\mathcal{H}om(\mathcal{F}; R\mathcal{H}om(G,H))$$
$$R\mathrm{Hom}(\mathcal{F} \overset{L}{\otimes} G, H) \cong R\mathrm{Hom}(\mathcal{F}, R\mathcal{H}om(G,H))$$
$$\mathrm{Hom}(\mathcal{F} \overset{L}{\otimes} G, H) \cong \mathrm{Hom}(\mathcal{F}, R\mathcal{H}om(G,H))$$

same proof idea as above.

This is the tensor-Hom adjunction we mentioned in the beginning.

Theorem 1.4.9. (Projection Formula)
Let $f: X \to Y$ be a continuous map of locally compact spaces, and assume k has finite global dimension. For $\mathcal{F} \in D^+(X,k)$ and $G \in D^+(Y,k)$, there is a natural isomorphism

$$f_!\mathcal{F} \overset{L}{\otimes} G \xrightarrow{\cong} f_!(\mathcal{F} \overset{L}{\otimes} f^*G)$$

the most significant result of this part of the talk is the projection formula

there is a subtelty in the proof of this theorem: the ab. category version is not true!

For sheaves the above map is not an iso, but it is an isomorphism if G is flat, and it is enough to carry out the proof on the level of derived categories.

Definition 1.4.20. Let $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$ be the projection maps. For $\mathcal{F} \in D^-(X,k)$ and $\mathcal{G} \in D^-(Y,k)$, their *external tensor product* $\mathcal{F} \boxtimes \mathcal{G}$ is the object

$$\mathcal{F} \boxtimes \mathcal{G} := pr_1^* \mathcal{F} \otimes pr_2^* \mathcal{G} \in D^-(X \times Y, k)$$

This defines a functor $\boxtimes : D^-(X,k) \times D^-(Y,k) \to D^-(X \times Y, k)$, and if k has finite global dimension: $\boxtimes : D^+(X,k) \times D^+(Y,k) \to D^+(X \times Y, k)$.

finally, we need to cover the external tensor product of sheaves.

Proposition 1.4.21 Let $f : X \to X'$ and $g : Y \to Y'$ be continuous maps.

the following proposition is easily deduced from the results above.

1) For $\mathcal{F} \in D^-(X',k)$, $\mathcal{G} \in D^-(Y',k)$, there is a natural isomorphism $f^* \mathcal{F} \boxtimes g^* \mathcal{G} \xrightarrow{\cong} (f \times g)^* (\mathcal{F} \boxtimes \mathcal{G})$.

~~If k has finite global dimension, same for D^+.~~

2) For $\mathcal{F}, \mathcal{F}' \in D^-(X,k)$, $\mathcal{G}, \mathcal{G}' \in D^-(Y,k)$, there is a natural isomorphism

$$(\mathcal{F} \overset{L}{\otimes} \mathcal{F}') \boxtimes (\mathcal{G} \overset{L}{\otimes} \mathcal{G}') \cong (\mathcal{F} \boxtimes \mathcal{G}) \overset{L}{\otimes} (\mathcal{F}' \boxtimes \mathcal{G}')$$

~~3) Assume k has finite global dimension and that X, Y are locally compact~~

---

3) Assume k has finite global dimension and that our spaces are locally compact.
For $\mathcal{F} \in D^+(X,k)$, $\mathcal{G} \in D^+(Y,k)$ there is a natural isomorphism

$$f_! \mathcal{F} \boxtimes f_! \mathcal{G} \cong (f \times g)_! (\mathcal{F} \boxtimes \mathcal{G}).$$

we have a corollary

Corollary 1.4.22. (Künneth formula) Let X, Y be locally comp, k-fin global dim. Then $R\Gamma_c(\underline{k}_X) \overset{L}{\otimes} R\Gamma_c(\underline{k}_Y) \cong R\Gamma_c(\underline{k}_{X \times Y})$
