Talk: Perverse Sheaves and Springer Theory

§1 The Springer resolution

Setup: $G \supseteq B \supseteq T$ conn red grp / Borel / max torus

Lie $\{$ $\mathfrak{g} \supseteq \mathfrak{b} \supseteq \mathfrak{h}$ red. Lie alg / Borel / Cartan

$W = N_G(T)/T$ Weyl grp

Example (different color!): $GL_n \supseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \supseteq \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

$\mathfrak{gl}_n \supseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \supseteq \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

$W = S_n$.

Def The nilpotent cone is $\mathcal{N} = \{ x \in \mathfrak{g} \mid x \text{ nilpotent (on } \mathfrak{g}\text{-reps)} \} \overset{\text{Ex}}{=} \{ x \in \mathfrak{g} \mid x \text{ nilp. matrix} \}$

Example (GL_2): $\mathcal{N} = \{ x \in \mathfrak{gl}_2 \mid \det(x) = \operatorname{tr}(x) = 0 \} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \;\middle|\; -a^2 - bc = 0 \right\}$ 

Fact
- $\mathcal{N}$ is irred
- $\dim \mathcal{N} = \dim G - \dim H = \dim \mathfrak{g} - \dim \mathfrak{h}$.
- $G \curvearrowright \mathcal{N}$ by adj action $\overset{\text{and}}{\Longrightarrow}$ $\mathcal{N}$ has fin. many G-orbits
- $\exists!$ dense G-orbit $\mathcal{O}_{reg} \subseteq \mathcal{N}$ (regular orbit), $\exists!$ closed G-orbit $\mathcal{O}_0 = \{0\}$.

Def Let $\mathcal{B} = G/B \cong \{\text{Borels in } G\}$ (Proof: • normalizer of Borel B is B • all Borels are conjugate)

$\mathfrak{g}_{rs} = \{ x \in \mathfrak{g} \mid x \text{ regular semisimple} \}$

Def Let $\tilde{\mathfrak{g}} = \{ (gB, x) \in \mathcal{B} \times \mathfrak{g} \mid x \in Ad(g)(\mathfrak{b}) \} \xrightarrow{\mu = pr_2} \mathfrak{g}$.

$\tilde{\mathcal{N}} = \mu^{-1}(\mathcal{N})$, $\tilde{\mathfrak{g}}_{rs} = \mu^{-1}(\mathfrak{g}_{rs})$

$\rightsquigarrow$ comm diag (by constr!)

$$\begin{tikzcd}
\tilde{\mathfrak{g}}_{rs} \arrow[r, hook] \arrow[d, "\mu_{rs}"'] & \tilde{\mathfrak{g}} \arrow[d, "\mu_{\mathfrak{g}}"] & \tilde{\mathcal{N}} \arrow[l, hook'] \arrow[d, "\mu_{\mathcal{N}}"] \\
\mathfrak{g}_{rs} \arrow[r, hook] & \mathfrak{g} & \mathcal{N} \arrow[l, hook']
\end{tikzcd}$$

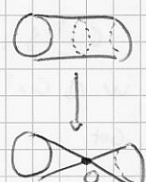
- $\mu_{\mathfrak{g}}$ = Grothendieck–Springer res.
- $\mu_{\mathcal{N}}$ = Springer resolution

Ex (GL_2): $G/B = GL_2 / \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \cong \mathbb{P}^1$.

$$\tilde{\mathcal{N}} = \left\{ \left([u:v], \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right) \;\middle|\; \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0,\ -a^2 - bc = 0 \right\}$$

$\big\downarrow \mu = pr_2$.

$\mathcal{N}$



<u>Proof</u> $x \in \mathcal{N} \cap Ad(g)(\mathfrak{b})$

$\Leftrightarrow x = g b g^{-1}$ with b nilp, i.e. $b = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$

Schubert cells: $\mathbb{P}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,\dot\cup\, \begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \overset{g=1}{x \in \mathcal{N}} \rightsquigarrow x = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \Rightarrow [u:v] = [1:0]$

$\begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow x = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = b \cdot \begin{pmatrix} -a & a^2 \\ -1 & a \end{pmatrix} \rightarrow [u:v] = [a:1]$.

Fiber over 0 is $\mathbb{P}^1 = G/B$!

Fiber over $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \neq 0$ is a single point (as $\dim \ker \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = 1$)! $\square$

Fact 1) $\mu: \tilde{\mathcal{N}} \to \mathcal{N}$ is smooth and proper. ([CG, Prop 3.1.34])
$\quad\;\; \tilde{\mathfrak{g}} \to \mathfrak{g}$

2) $\tilde{\mathcal{N}} \cong T^* G/B$ ([CG, Lemma 3.2.2])

§2 The Springer (perverse) sheaf

Lemma 1) $\mu: \tilde{\mathcal{N}} \to \mathcal{N}$ is semismall (wrt the stratification by G-orbits)

2) $\mu: \tilde{\mathfrak{g}} \to \mathfrak{g}$ is small wrt $\mathfrak{g}_{rs} \subseteq \mathfrak{g}$.

Proof Only 1), 2) is very similar (see [Achar 8.2.5]).
in detail

Let $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ Steinberg variety $= \{(g_1 B, g_2 B, x) \in \mathcal{B} \times \mathcal{B} \times \mathcal{N} \mid x \in Ad(g_1)(\mathfrak{b}) \cap Ad(g_2)(\mathfrak{b})\}$

Fact $f: X \to Y$ proper is semismall $\iff \dim X \times_Y X \leq \dim X$

$\rightsquigarrow$ suff to show $\dim Z \leq \dim \mathcal{N}$.

$w \in W \rightsquigarrow$ let $Z_w = \{(g_1 B, g_2 B, x) \in Z \mid g_1^{-1} g_2 B \in BwB\}$ (Schubert cell of $G/B = \mathcal{B}$)

$\Rightarrow Z = \dot\bigcup_{w \in W} Z_w$, Z_w loc closed $\rightsquigarrow$ suff to show $\dim Z_w \leq \dim \mathcal{N}$ $\forall w$.

Fix $w \in W$ (repr. $\dot w \in N_G(T)$). Note: $Z_w \cong G \times^{B \cap \dot w B \dot w^{-1}} (\mathfrak{n} \cap Ad(\dot w)\mathfrak{n})$
$(gB, g\dot w B, Ad(g)(x)) \leftarrow (g, x)$

$\Rightarrow \dim Z_w \leq \dim G - \dim (B \cap \dot w B \dot w^{-1}) + \dim(\mathfrak{n} \cap Ad(\dot w)\mathfrak{n})$

Note: $Lie(B \cap \dot w B \dot w^{-1}) = \mathfrak{h} \oplus (\mathfrak{n} \cap Ad(\dot w)(\mathfrak{n}))$

$\Rightarrow \dim Z_w \leq \dim G - \dim \mathfrak{h} - \dim(\mathfrak{n} \cap Ad(\dot w)(\mathfrak{n})) + \dim(\mathfrak{n} \cap Ad(\dot w)\mathfrak{n})$
$= \dim G - \dim \mathfrak{h} = \dim \mathcal{N}$.

For 2): $Z' = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} = \dot\bigcup_{w \in W} Z'_w$, $Z'_w \cong G \times^{B \cap \dot w B \dot w^{-1}} (\mathfrak{b} \cap Ad(\dot w)(\mathfrak{b}))$ $\rightsquigarrow \dim Z'_w = \dim \mathfrak{g}$

$Z'' = \{(g_1 B, g_2 B, x) \in Z' \mid x \notin \mathfrak{g}_{rs}\}$, $Z''_w = Z'_w \cap Z''$

$Z''_w \cong G \times^{B \cap \dot w B \dot w^{-1}} (\mathfrak{b} \cap Ad(\dot w)\mathfrak{b} \cap (\mathfrak{g} \setminus \mathfrak{g}_{rs}))$

$\emptyset \neq \mathfrak{g}_{rs} \subseteq \mathfrak{g}$ open $\Rightarrow \dim \mathfrak{g} \setminus \mathfrak{g}_{rs} < \dim \mathfrak{g} \Rightarrow \dim Z''_w < \dim Z'_w = \dim \mathfrak{g}$ $\forall w$.

Rest follows by choosing an appropriate stratification of $\mathfrak{g}_{rs}$ (fiber dim thm). □

Lemma $\mu_{rs}: \tilde{\mathfrak{g}}_{rs} \to \mathfrak{g}_{rs}$ is a covering map with deck transforms $\cong W$

Proof $\mathfrak{b}_{rs} = \mathfrak{b} \cap \mathfrak{g}_{rs}$; $\mathfrak{h}_{rs} = \mathfrak{h} \cap \mathfrak{g}_{rs}$ $\rightsquigarrow$ $\tilde{\mathfrak{g}}_{rs} = G \times^B \mathfrak{b}_{rs}$ $\cancel{\leftarrow G \times \mathfrak{h}_{rs}}$. Get

$G \times^T \mathfrak{h}_{rs} \longrightarrow G \times^B \mathfrak{b}_{rs} = \tilde{\mathfrak{g}}_{rs}$ bijection $\Rightarrow$ iso (Zariski main thm, $\tilde{\mathfrak{g}}_{rs}$ normal)

$W \circlearrowright G \times^T \mathfrak{h}_{rs}$ by $w \cdot (gT, x) = (g \dot w^{-1} T, Ad(\dot w)(x))$ free action

. Get
$$\begin{tikzcd} G \times^T \mathfrak{h}_{rs} \arrow[r] \arrow[d] & G \times^B \mathfrak{b}_{rs} \cancel{\cong \mathfrak{g}_{rs}} \arrow[d, "\mu"] \\ (G \times^T \mathfrak{h}_{rs})/W \arrow[r, "\exists !"', "\bar\mu"] & \mathfrak{g}_{rs} \end{tikzcd}$$

Claim $\bar\mu$ is an iso. (suff to check bij: see above)

But for $x \in \mathfrak{g}_{rs}$, $|\mu^{-1}(x)| = |\{gB \in \mathcal{B} \mid x \in Ad(g)(\mathfrak{b})\}| = |W|$ as reg. ss. x lies in ~~precisely~~ a unique Cartan, and each Cartan lies in $|W|$ Borels.

Def The Springer sheaf is $Spr = \mu_* \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$

Lemma $Spr \in Perv(\mathcal{N})$ is semisimple.

Proof ~~Clear~~ $\mu: \tilde{\mathcal{N}} \to \mathcal{N}$ is semismall. $\Rightarrow \mu_* \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$ perverse; Semisimple by Decomp Thm. □

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Here $L_{rs} : \mathbb{C}[W]\text{-mod}_{fg} \longrightarrow \mathbb{C}[\pi_1(\mathfrak{g}_{rs}, *)]\text{-mod}_{fg} \xrightarrow{\sim} \mathrm{Loc}_{ft}(\mathfrak{g}_{rs}, \mathbb{C})$

<u>Lemma</u> (Cor of prev. Lemma):

1) $\nu_{rs *} \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{rs}} \cong L_{rs}(\mathbb{k}[W])$

2) $\nu_* \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}] = IC(\mathfrak{g}_{rs}, \nu_{rs *} \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{rs}})$

3) $\mu_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}] = \mathrm{incl}_{\mathcal{N}}^* IC(\mathfrak{g}_{rs}, \nu_{rs *} \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{rs}})[\text{-}\dim T]$

<u>Proof</u> 1) From interaction of monodromy + covering spaces

2) From 1) + [Achar, Prop 3.8.7]

3) From 2) + proper base change

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[r, hook, "i_{\tilde{\mathcal{N}}}"] \arrow[d, "\mu"'] & \tilde{\mathfrak{g}} \arrow[d, "\nu"] \\ \mathcal{N} \arrow[r, hook, "i_{\mathcal{N}}"'] & \mathfrak{g} \end{tikzcd}$$

$\square$

<u>Cor</u> $\mathrm{End}(\mathrm{Spr}) \cong \mathbb{C}[W]$

<u>Proof</u> $\mathrm{End}(\mu_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]) \cong \mathrm{End}(IC(\mathfrak{g}_{rs}, \nu_{rs *} \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{rs}}))$

$\cong \mathrm{End}(\nu_{rs *} \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{rs}})$

$\cong \mathrm{End}_{\mathbb{C}[W]}(\mathbb{C}[W]) \cong \mathbb{C}[W]$ $\square$

<u>Cor</u> $W \curvearrowright H^\bullet(\tilde{\mathcal{N}})$

<u>Proof</u> $\mathrm{End}(\mu_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]) \cong \mathrm{End}(\underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]) \cong H^\bullet(\tilde{\mathcal{N}})$ (up to shift) $\square$

<u>Rem</u>
- This action is something that doesn't come from $W \curvearrowright \tilde{\mathcal{N}}$! there is no (reasonable) action of W on $\tilde{\mathcal{N}}$!
- this can be used to build irreps of W (Springer correspondence)

<u>Ex</u> Springer sheaf for $SL(2)$:

$\tilde{\mathcal{N}} = \{ ([u:v], \begin{pmatrix} a & b \\ c & -a \end{pmatrix}) \mid \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0, -a^2 - bc = 0 \}$

$\mathcal{N} = \{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid -a^2 - bc = 0 \} = \{0\} \cup (\mathcal{N} \setminus \{0\})$. <u>Fact</u> $\pi_1(\mathcal{N} \setminus \{0\}) = \mathbb{Z}/2\mathbb{Z}$.

- simple perv. sheaves on $\mathcal{N}$:

(~~stalks of~~ cohomologies on the strata)

$IC(\{0\}, \underline{\mathbb{C}}) =$

	U	$\{0\}$
0		$\underline{\mathbb{C}}$

$IC(U, \mathrm{triv}) =$

	U	$\{0\}$
-2	$\underline{\mathbb{C}}$	$\underline{\mathbb{C}}$

$IC(U, \mathrm{sgn}) =$

	U	$\{0\}$
-2	sgn	

- $\mu_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}] =$

	U	$\{0\}$
0		$\underline{\mathbb{C}}$
-1		
-2	$\underline{\mathbb{C}}$	$\underline{\mathbb{C}}$

$\Rightarrow \mathrm{Spr} = IC(\{0\}, \underline{\mathbb{C}}) \oplus IC(U, \mathrm{triv.})$.