Perverse Sheaves Seminar Talk 14

Part I

let us first recall the notions of small and semi-small morphisms introduced in talk 12.

Def 1 (3.8.1 (Achar)) A morphism of varieties $f: X \to Y$ is said to be semi-small if Y admits a stratification $(Y_t)_{t \in \mathcal{T}}$ such that for each stratum Y_t and each point $y \in Y_t \cap f(X)$, we have;

$$\dim f^{-1}(y) \leq \frac{1}{2}(\dim X - \dim Y_t)$$

the morphism is said to be small with respect to an open dense $W \subset Y$ if;

(1) For each $y \in W$, $f^{-1}(y)$ is a finite set

(2) There exists a stratification $(Y_t)_{t \in \mathcal{T}}$ of Y such that W is a union of strata and for each strata $Y_t \subset Y \setminus W$ and $y \in Y_t \cap f(X)$ we have

$$\dim f^{-1}(y) \leq \frac{1}{2}(\dim X - \dim Y_t)$$

These class of morphisms are important as they repair defects of t-exactness for proper morphisms.

Theorem 2 let $f: X \to Y$ proper semismall with X smooth, connected then $f_* \mathcal{L}[\dim X]$ is perverse on Y

let us give some intuition as to why the dimensionality conditions are important. A crucial lemma that we will prove soon is that if Z is proper $H^{2d}(Z,\mathbb{C}) \neq 0$ ($d = \dim Z$).

let $f: \overset{\text{smooth}}{X} \to Y$ proper, $W \subset Y$ subvariety with fiber dimension $\geq d$

$$
\begin{tikzcd}
\dim \geq d = f^{-1}(w) \arrow[r] \arrow[d] & f^{-1}(W) \arrow[r] \arrow[d] & X \arrow[d, "f \text{ proper}"] \\
w \arrow[r] & W \arrow[r] & Y
\end{tikzcd}
$$

By proper base change and the above lemma we have

$$\dim \operatorname{supp} H^{2d - \dim X} f_* \mathbb{C}[\dim X] \geq \dim W$$

so if we want $f_* \mathbb{C}[\dim X]$ to be in ${}^{p}D_c^{\leq 0}(Y)$ then we must require;

$$\dim W \leq \dim \operatorname{supp} H^{2d - \dim X} f_* \mathbb{C}[\dim X] \leq -2d + \dim X$$

$$\dim W \leq -2d + \dim X$$

$$2 \dim f^{-1}(w) \leq -\dim W + \dim X$$

which is precisely the inequality involved in the definition of semi-small morphisms. Theorem 2 then tells us this is sufficient.

Lemma 3 If X is a proper variety of dimension d then $H^{2d}(X, \mathbb{C}) \neq 0$.

(Convince yourself that this does not contradict Artin vanishing (Hint: what does proper + affine = ?)

Proof Verdier duality gives $R\mathrm{Hom}_{D(Y)}(i_! F, G) \cong R\mathrm{Hom}_{D(X)}(F, i^! G)$

taking global sections gives $\mathrm{Hom}_{D(Y)}(i_! F, G) \cong \mathrm{Hom}_{D(X)}(F, i^! G)$

specializing to $F = \mathbb{C}[-2d]$ and $G = \mathbb{C}$ and i map to the point.

$\mathrm{Hom}_{D(k)}(i_! \mathbb{C}[-2d], \mathbb{C})$ but $i_! \mathbb{C}[-2d] = H^{2d}(X, \mathbb{C})$ as X is proper

$\cong H^{2d}(X, \mathbb{C})^\vee$ (dual in the derived category)

OTOH we have

$\mathrm{Hom}_{D^+(X,k)}(\mathbb{C}[-2d], i^! \mathbb{C})$ by general thing

$\cong \mathrm{Ext}^{-2d}(\mathbb{C}, i^! \mathbb{C})$

$\cong H^{-2d}(X, \omega_X)$ all in all we have

$H^{2d}(X, \mathbb{C})^\vee \cong H^{-2d}(X, \omega_X)$

$\therefore$ $H^{2d}(X, \mathbb{C}) \neq 0 \Rightarrow H^{-2d}(X, \omega_X) \neq 0$

Now take a smooth open subvariety $j : \mathcal{U} \to X$ with complement $i : Z \to X$ with $\dim Z < d$ we have a distinguished triangle

$$i_* W_Z \to W_X \to j_* W_{\mathcal{U}} \to \cdots$$

apply H^0 to get a LES

$$H^{-2d}(Z, W_Z) \to H^{-2d}(X, W_X) \to H^{-2d}(\mathcal{U}, W_{\mathcal{U}}) \to H^{-2d+1}(Z, W_Z)$$

Since $\dim Z < d$, by what we have shown previously and the grothendieck vanishing theorem, we get,

$$H^{-2d}(Z, W_Z) = H^{-2d+1}(Z, W_Z) = 0$$

Therefore $H^{-2d}(X, W_X) \cong H^{-2d}(\mathcal{U}, W_{\mathcal{U}})$. Since there is atleast one connected component of $\mathcal{U}$ with $\dim d$, $\mathcal{U}_0$;

$W_{\mathcal{U}_0} = \underline{\mathbb{C}}[2d]$ so

$$H^{-2d}(\mathcal{U}_0, W_{\mathcal{U}_0}) = H^0(\mathcal{U}_0, \underline{\mathbb{C}}) = \mathbb{C} \neq 0 \qquad \square$$

<u>Def 4</u> let X be a variety with a good stratification $(X_s)_{s \in S}$
$f : X \to Y$ is called <u>stratified semi-small</u> if $\forall s \in S$ $f|X_s$ is semi-small.
f is called <u>stratified small</u> w.r.t W if $\forall s \in S$ $f|X_s$ is small with respect to W.

<u>Theorem 5</u> Let $f : X \to Y$ be a proper, stratified semi-small morphism. Then $f_* \, D^b_S(X, \mathbb{K}) \to D^b_C(Y, \mathbb{K})$ is t-exact. ~~for~~

Theorem 6

let $f: X \to Y$ be a proper stratified small morphism wrt W. let $f' = f|_{f^{-1}(W)}$, $h: W \hookrightarrow Y$ Then for any perverse sheaf F, there is a natural isomorphism

$$f_* F \cong h_{!*} f'_* (F|_{f^{-1}W})$$

Let us make some quick definitions that will lead us to the main theorem of this first part.

Def 7 let X be a variety. An object in $D^b_c(X, \mathbb{Q})$ is said to be a semi-simple complex if it is isomorphic to a finite direct sum of shifts of perverse sheaves. The additive category of semi-simple complexes is denoted $\mathrm{Semis}(X, \mathbb{Q})$

Theorem 8 Let $f: X \to Y$ be a proper morphism of varieties. For any $F \in \mathrm{Semis}(X, \mathbb{Q})$ we have $f_* F \in \mathrm{Semis}(Y, \mathbb{Q})$

Rmk As an important special case, if X is smooth, the constant sheaf $\underline{\mathbb{Q}}_X$ is a semi-simple complex, so $f_* \underline{\mathbb{Q}}_X$ is semi-simple. If in addition f is semi-small, $f_* \underline{\mathbb{Q}}_X[\dim X]$ is a semi-simple perverse sheaf.

Now we will move onto the main example for this part. Recall the Schubert stratification for $Gr(d,n)$ Fix a flag $E_1 \subset \dots \subset E_n \cong \mathbb{C}^n$ with $\dim E_q = q$. given a partition λ $(n-d \geq \lambda_1 \geq \lambda_2 \dots \lambda_d \geq 0)$ The schubert cell Ω°_λ is the set of subspaces $\{F \subset \mathbb{C}^n \mid \dim(F \cap E_q) = k$ for $q \in [n-d+k-\lambda_k, n-d+k-\lambda_{k+1}]$, $k = (0 \dots d)\}$

Then we have that the schubert cells are disjoint and

$$Gr(d,n) = \bigsqcup_{\lambda} \Omega_{\lambda}$$

In a more general setting let G be a semi-simple algebraic group, fix a Borel B, $T \subset B$ max torus. G/B is called the <u>flag variety</u> and it has a similar decomposition

$$G/B = \bigsqcup_{w} B\dot{w}B/B$$ where $\dot{w}$ is a lift of $w \in W$ $W = \frac{N_G(T)}{T}$

To see why this is called the flag variety, notice that GL_n acts transitively on the set of flags, and the stabilizer of the standard flag is the set of all upper triangular matricies.

Call $B\dot{w}B/B =: \mathcal{O}_w$ and $\overline{\mathcal{O}_w} = X_w$

we will now switch to a more symmetric setting with $G/B \times G/B$

$$G/B \times G/B = \bigsqcup_{w} O_w := \bigsqcup_{w} G \cdot (B/B, \dot{w}B/B)$$

the old cells were B orbits, the new cells are G orbits.

Denote $\mathcal{X}_w = \overline{O_w}$

the variety $\mathcal{X}_w$ (resp $\mathcal{O}_w$) is fibered over G/B with fiber X_w (resp O_w)

$$\begin{array}{ccc} (gB/B,\ gwB/B) & \text{but also} & (gbB/B,\ gbwB/B) \\ \downarrow & & \downarrow \\ gB/B & & gB/B \end{array}$$

for any $b \in B \Rightarrow BwB$ is the fiber.

<u>Example</u> $SL(n)$, G/B flag variety

$\mathbb{C}^n = V_n \supset V_{n-1} \cdots V_0 = \{0\}$ $\dim V_i = i$

$W = S_n$ symmetric group.

$(V_\cdot, V_\cdot')$ $\in O_w$ if there exists a basis
$(G/B \times G/B)$ of $\mathbb{C}^n = \{e_1 \ldots e_n\}$ st

$V_i = \mathrm{Span}(e_1 \ldots e_i)$, V_i' $\mathrm{Span}(e_{w(1)} \ldots e_{w(i)})$

We now consider the variety, called the Bott-Samelson variety

$\tilde{\mathcal{X}}_{\underline{w}} := G/B \underset{G/P_{s_1}}{\times} G/B \cdots \underset{G/P_{s_k}}{\times} G/B$ where P_{s_i} is the parabolic subgroup associated to the simple reflection s_i in the decomposition $w = s_1 \ldots s_k$

It has an explicit description

$$\tilde{\mathcal{X}}_{\underline{w}} = \{x_1 \ldots x_{k+1} \in (G/B)^{k+1} \mid (x_\ell, x_{\ell+1}) \in \mathcal{X}_{s_{i_\ell}}\}$$

<u>Rmk</u> $\tilde{\mathcal{X}}_{\underline{w}}$ is smooth

- $$\begin{tikzcd} \tilde{\mathcal{X}}_{\underline{s_1}} \arrow[r] \arrow[d] & G/B \arrow[d] \\ G/B \arrow[r] & G/P_{s_1} \end{tikzcd}$$
  ← surjective $\mathbb{P}^1$ fibration
  use miracle flatness, and use projectiveness to see properness.

$\Rightarrow$ $\tilde{X}_{s_1}$ flat with smooth fibers over smooth variety G/B as G/B is a group.

- The fact that $G/B \to G/P_s$ is a $\mathbb{P}^1$ fibration follows from
  $$\begin{tikzcd} P/B \arrow[r] \arrow[d] & G/B \arrow[d] \\ * \arrow[r] & G/P \end{tikzcd}$$

and it is a representation theory fact that $P/B \cong \mathbb{P}^1$

- To get a more intuitive picture for the case of GL_n, notice that just like how G/B parametrizes complete flags, G/P parametrizes partial flags. In the case of P_{s_1}; $\mathbb{C}^n = V_{n-1} \cdots V_1 = \mathbb{C}^2$, $V_0 = \{0\}$.

Also note by the same argument from the projectivity of G/B, G/P we see that the map

- $\pi_{\underline{w}} : \tilde{\mathcal{X}}_{\underline{w}} \to \mathcal{X}_w$ given by $(x_1 \cdots x_{k+1}) \to (x_1, x_{k+1})$ is proper

In the case of SL_3, $W = S_3$ generated by two reflections (s_1, s_2) $W = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

In the Weyl group we have the bruhat order where $x < y$ if the reduced expression for x can be obtained from that of y bey deleting reflections. the Bruhat order translates to the inclusion of Schubert varieties.

let $w = s_1 s_2$ The Bott-Samelson variety consists of triples of flags $(V_\bullet^{(1)}, V_\bullet^{(2)}, V_\bullet^{(3)})$ satisfying $V_2^{(1)} = V_2^{(2)}$ and $V_1^{(2)} = V_1^{(3)}$ so $V_\bullet^2$ is completely determined by $V_\bullet^1, V_\bullet^3$ and $\pi_{\underline{w}}$ is an isomorphism. In particular $\mathcal{X}_w$ is smooth. Similarly for $\tilde{\mathcal{X}}_{s_2 s_1}$

let $\underline{w} = s_1 s_2 s_1$ $\tilde{\mathcal{X}}_{\underline{w}}$ consists of 4-tuples of flags $(V_\bullet^{(1)}, V_\bullet^{(2)}, V_\bullet^{(3)}, V_\bullet^{(4)})$ satisfying

$$V_2^{(1)} = V_2^{(2)}, \quad V_1^{(2)} = V_1^{(3)}, \quad V_2^{(3)} = V_2^{(4)}$$

so $V_\bullet^2$ is completely determined by $V_\bullet^{(1)}, V_\bullet^{(3)}$ so thus simplifies to triples $(V_\bullet^{(1)}, V_\bullet^{(2)}, V_\bullet^{(3)})$ with

$$V_1^{(2)} \subset V_2^{(1)}, \quad V_2^{(2)} = V_2^{(3)} \qquad \text{so}$$

$V_2^{(2)}$ is determined by $V_\bullet^3$. Since $V_1^{(2)} \subset V_2^{(1)} \cap V_2^{(3)}$ it is also determined if $V_2^{(1)} \neq V_2^{(3)}$

Otherwise, when $V_2^{(1)} = V_2^{(3)}$ which is the same as saying $(V_\bullet^1, V_\bullet^3) \in \mathcal{O}_{s_1}$, we have $\mathbb{P}(V_2^{(1)}) = \mathbb{P}^1$ choices.

To summarize the fiber over $\mathcal{O}_{s_1}$ is isomorphic to $\mathbb{P}^1$ and over $G/B \times G/B \setminus O_{s_1}$ it is isomorphic to a point.

Note that $\dim O_{s_1} = 4$ as it is a fibration over B with fiber $\mathcal{O}_{s_1}$ and $\dim \mathcal{O}_{s_1} = 1$. This is due to the correspondance between bruhat orders and schubert varieties, where the length of the word is the dimension of the cell.

Since $\dim G/B \times G/B = 6$ and $2 + 4 \leq 6$, π_w is semi-small.

We will now introduce $\mathcal{H}(W)$ as the free $\mathbb{Z}[v, v^{-1}]$ module with basis T_w, $w \in W$. Let $\mathcal{S}$ denote the Bruhat stratification. For $F \in D^b_{\mathcal{S}}(G/B \times G/B)$ write

$$h(F) = \sum_{w \in W} \left(\sum_{i \in \mathbb{Z}} \dim H^{-i}(F_w) v^i \right) T_w \in \mathcal{H}(W)$$

where F_w is the fiber of F at the point (B, wB). It is a complex of finite vector spaces so cohomology and dimension are well defined.

We have just computed;

$$h(\pi_{s_1s_2*}(\mathbb{C}_{\tilde{\mathfrak{X}}_{s_1s_2}})) = T_{s_1s_2} + T_{s_1} + T_{s_2} + T_1$$

$$h(\pi_{s_1s_2s_1*}(\mathbb{C}_{\tilde{\mathfrak{X}}_{s_1s_2s_1}}[6])) = v^6(T_{s_1s_2s_1} + T_{s_1s_2} + T_{s_2s_1} + T_{s_2})$$
$$(v^6+v^4)(T_{s_1} + T_1)$$

This can be seen through proper base change and considering the cohomology of projective space.

Since $\mathfrak{X}_{s_1s_2s_1}$, $\mathfrak{X}_{s_1}$ are both smooth; the first because it is the product of two flag varieties, the second can be seen through computing its Bott-Samelsen variety and showing it is an iso. We therefore have

$$h(IC(\mathfrak{X}_{s_1}, \mathbb{C})) = v^4(T_s + T_1)$$
$$h(IC(\mathfrak{X}_{s_1s_2s_1}, \mathbb{C}) = v^6(T_{s_1s_2s_1} + T_{s_1s_2} + T_{s_2s_1} + T_{s_1} + T_{s_2} + T_1)$$

The decomposition theorem tells us that $\pi_{s_1s_2s_1*}\,\underline{\mathbb{C}}_{\mathfrak{X}_{s_1s_2s_1}}[6]$ must be perverse semi-simple. Furthermore we have the fact* that the Hecke algebra $H(W)$ is isomorphic to the split Grothendieck group of the category of semi-simple complexes. and thus we have show

$$\pi_{s_1s_2s_1*}\,\mathbb{C}_{\tilde{\mathfrak{X}}_{s_1s_2s_1}}[6] \cong IC(\mathfrak{X}_{s_1s_2s_1}) \oplus IC(\mathfrak{X}_{s_1})$$

* Achar 7.3.8