# Perverse Sheaves IV: t-exactness under affine pushout and smooth pullback

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## Contents



## 1 t-Exactness under Morphism with Bounded Fiber Dimensions

In this note, we consider only varieties $X$ or $Y$ over the field $\mathbb{C}$, and the coefficient ring of local systems and constructible sheaves are also $\mathbb{C}$. We write $\mathrm{H}^i(A)$ for the i-th cohomology object of a chain complex $A$. We write $\mathbf{H}^i(X, \mathcal{F})$ for the $i$ -th sheaf (hyper) homology of a topological space X with coefficients in a sheaf (or chain complex of sheaves) $\mathcal{F}$.

In a previous discussion (talk 10), we explored the perverse t-structure on $D_c^{\mathrm{b}}(X, \mathbb{C})$ and the behavior of perversity under immersions. Specifically, we proved the following lemma:

**Lemma 1.1.** *Let $j : U \to X$ be an open embedding, and let $i : Z \to X$ be a closed embedding.*

*(1) $j^*$ is t-exact, i.e., $\mathrm{j}^*({}^{\mathrm{p}}\mathrm{D}_c^{\geq 0}(X)) \subseteq {}^{\mathrm{p}}\mathrm{D}_c^{\geq 0}(U)$ and $\mathrm{j}^*({}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(X)) \subseteq {}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(U)$.*

*(2) $\mathrm{j}_!$ is right t-exact, i.e., $\mathrm{j}_!({}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(U)) \subseteq {}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(X)$.*

*(3) $\mathrm{j}_*$ is left t-exact, i.e., $\mathrm{j}_*({}^{\mathrm{p}}\mathrm{D}_c^{\geq 0}(U)) \subseteq {}^{\mathrm{p}}\mathrm{D}_c^{\geq 0}(X)$.*

*(4) $\mathrm{i}_*$ is t-exact, i.e., $\mathrm{i}_*({}^{\mathrm{p}}\mathrm{D}_c^{\geq 0}(X)) \subseteq {}^{\mathrm{p}}\mathrm{D}_c^{\geq 0}(Z)$ and $\mathrm{i}_*({}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(X)) \subseteq {}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(Z)$.*

*(5) $\mathrm{i}^*$ is right t-exact, i.e., $\mathrm{i}^*({}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(X)) \subseteq {}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(Z)$.*

*(6) $\mathrm{i}^!$ is left t-exact, i.e., $\mathrm{i}^!({}^{\mathrm{p}}\mathrm{D}_c^{\geq 0}(X)) \subseteq {}^{\mathrm{p}}\mathrm{D}_c^{\geq 0}(Z)$.*

In the current discussion, we aim to find the behavior of perversity under functors defined by morphisms with bounded fiber dimensions. We will establish the t-exactness of smooth pullback and quasi-finite affine pushforward. The primary reference is chapters 3.5-3.6 of [Ach21]. The main theorem is as follows.

**Theorem 1.2.** *Let $f$ be a morphism with fibers of dimension $\leq d$, then*

*(a) $f^*[d]$ and $f_![d]$ are right t-exact for the perverse t-structures on $\mathrm{D}_c^{\mathrm{b}}(X)$ and $\mathrm{D}_c^{\mathrm{b}}(Y)$,*

*(b) $f^![-d]$ and $f_*[-d]$ are left t-exact for the perverse t-structures on $\mathrm{D}_c^{\mathrm{b}}(X)$ and $\mathrm{D}_c^{\mathrm{b}}(Y)$,*

*(c) The functors $({}^{\mathrm{p}}\mathsf{H}^d f^*, {}^{\mathrm{p}}\mathsf{H}^{-d} f_*)$ and $({}^{\mathrm{p}}\mathsf{H}^d f_!, {}^{\mathrm{p}}\mathsf{H}^{-d} f^!)$ form adjoint pairs between $\mathsf{Perv}(X)$ and $\mathsf{Perv}(Y)$.*

Consequently, we immediately affirm that a smooth pullback is t-exact, as $f^* \cong f^![-2d]$.

**Corollary 1.3.** *([Ach21]* Proposition *3.6.1 and* Lemma *3.6.2)*
*Suppose $f$ is smooth of relative dimension $d$. Then $f^\dagger := f^*[d] = f^![-d]$ is t-exact. The functor ${}^{\mathrm{p}}\mathsf{H}^{-d} f_*$ serves as its right adjoint.*

To establish the main theorem, it is essential to thoroughly analyze the formal implications of adjoint functors and t-structures within the triangulated category. Once we have done this, we can streamline proving the main theorem by demonstrating the right t-exactness of $f^*[d]$.

The first lemma asserts that taking the heart, the left-exact (or right-exact) functor in the triangulated category becomes a left-exact (or right-exact) functor in the Abelian category.

**Lemma 1.4.** *([Ach21]* Lemma A 7,14*)*
*Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulated categories equipped with t-structures. And let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote their hearts respectively. Let $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a triangulated functor.*

*(1) If $F$ is left t-exact, then the functor ${}^{\mathrm{t}}\mathsf{H}^0 \circ F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is left exact.*

*(2) If $F$ is right t-exact, then the functor ${}^{\mathrm{t}}\mathsf{H}^0 \circ F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is right exact.*

The following lemma addresses the relationship between the adjoint pair and the t-exactness, serving as a broadened version of the connection between adjoint pairs and exactness, specifically within the Abelian category.

**Lemma 1.5.** *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are triangulated categories equipped with t-structures, and a triangulated functor $G\colon \mathcal{T}_1 \to \mathcal{T}_2$ is left adjoint to another triangulated functor $F\colon \mathcal{T}_2 \to \mathcal{T}_1$, then $G$ is right t-exact if and only if $F$ is left t-exact.*

*Proof.* Given that $G$ is right t-exact, we can deduce that for any $\mathcal{F} \in \mathcal{T}_1^{\leq -1}$, we have $G(\mathcal{F}) \in \mathcal{T}_2^{\leq -1}$. Let's consider $\mathcal{G} \in \mathcal{T}_2^{\geq 0}$. Then, for any $\mathcal{F} \in \mathcal{T}_1^{\leq -1}$, we can establish the following hom-set bijection using adjointness:

$$\mathrm{Hom}_{\mathcal{T}_2}(\mathcal{F}, F\mathcal{G}) \cong \mathrm{Hom}_{\mathcal{T}_1}(G\mathcal{F}, \mathcal{G}) \cong 0,$$

and the t-structure axiom implies the last equality. Consequently, using the Lemma [Ach21]A.7.3, we get $F\mathcal{G} \in \mathcal{T}_1^{\geq 0}$.

Comparable statements can demonstrate the opposite direction. □

Upon investigating t-exactness under Verdier duality, one can derive it through a formal argument involving a dualizing functor. A dualizing functor denoted as $D$ is a contravariant functor, which interchanges the connective ($\leq 0$) and co-connective ($\geq 0$) parts of a given t-structure with the property that $D^2 = id$.

**Lemma 1.6.** *Let $\mathcal{T}_i$ $(1 \leq i \leq 2)$ represent the triangulated categories as described previously, and let $D_i$ denote the dualizing functors on $\mathcal{T}_i$, For a triangulated functor $F\colon \mathcal{T}_2 \to \mathcal{T}_1$, the composition $D_1 \circ F \circ D_2$ is right (left) t-exact if and only if $F$ is left (right) t-exact.*

*Proof.* Upon the assumption that $D_1 \circ F \circ D_2$ is right (left) t-exact, the objective is to demonstrate that for any $\mathcal{G} \in \mathcal{T}_2^{\geq 0}$, $F\mathcal{G} \in \mathcal{T}_1^{\geq 0}$. This is equivalent to proving that for any $\mathcal{F} \in \mathcal{T}_2^{\leq -1}$, $\mathrm{Hom}_{\mathcal{T}_1}(\mathcal{F}, F\mathcal{G}) \cong 0$ using the lemma [Ach21]A.7.3. Leveraging the property $D^2 = id$, we have

$$\mathrm{Hom}_{\mathcal{T}_1}(\mathcal{F}, F\mathcal{G}) \cong \mathrm{Hom}_{\mathcal{T}_1}(D_1^2\mathcal{F}, D_1^2 F\mathcal{G}) = D_1(\mathrm{Hom}_{\mathcal{T}_1}(D_1 F\mathcal{G}, D_1\mathcal{F})) = D_1(\mathrm{Hom}_{\mathcal{T}_1}((D_1 F D_2)(D_2\mathcal{G}), D_1\mathcal{F})), \tag{1}$$

It should be noted that $D_2\mathcal{G} \in \mathcal{T}_2^{\leq 0}$. Consequently, we infer that $(D_1 F D_2)(D_2\mathcal{G}) \in \mathcal{T}_1^{\leq 0}$ based on the right t-exactness property of $D_1 \circ F \circ D_2$. Considering that $D_1\mathcal{F} \in \mathcal{T}_1^{\geq 1}$, the equation (1) $= 0$ is an outcome of the axiom of t-structure.

Similar assertions can demonstrate the opposite direction. □

We can begin proof of the main theorem 1.2.

*Proof of Theorem 1.2.* By directly utilizing lemmas, we can complete the task by demonstrating that the functor $f^*[d]$ is right t-exact. Assuming the right t-exactness of $f^*[d]$, we can infer that its right adjoint functor, $f_*[-d]$, exhibits left t-exactness by Lemma 1.5. Furthermore, using Lemma 1.6 facilitates the demonstration that $f^![-d] \cong \mathbb{D}f^*[d]\mathbb{D}$ is left t-exact, thus establishing the validity of statement (b). The adjunct $f_![d] \dashv f^![-d]$ suggests that $f_![d]$ is right t-exact as per Lemma 1.5, thus substantiating statement (a). The application of Lemma 1.4 allows us to validate statement (c).

To show that the functor $f^*[d]$ is right t-exact, we begin by noting that, by the assumption on fibers, for any locally closed variety $Z \in Y$, it holds that $\dim f^{-1}(Z) \leq \dim Z + d$. Consequently, for a constructible sheaf $\mathcal{F}$ on $Y$, we have

$$\dim \operatorname{supp} f^*\mathcal{F} \leq \dim \operatorname{supp} \mathcal{F} + d,$$

Moreover, for $\mathcal{F} \in {}^{\mathrm{p}}\mathrm{D}^{\mathrm{b}}_c(Y, \mathbb{C})^{\leq 0}$, where $\dim \operatorname{supp} \mathrm{H}^i(\mathcal{F}) \leq -i$ for any $i$, we find that

$$\dim \operatorname{supp} \mathbf{H}^i(f^*\mathcal{F}[d]) = \dim \operatorname{supp} f^*(\mathbf{H}^i(\mathcal{F}[d]) \leq -(i+d) + d = -i,$$

Consequently, it follows that the functor $f^*[d]$ is right t-exact. □

## 2 t-Exactness under Affine Morphism

We aim to demonstrate that an affine pushforward is the right t-exact for perverse t-structure. The primary theorem is as follows:

**Theorem 2.1.** *Let $f\colon X \to Y$ be an affine morphism. Then the functor $f_*\colon \mathrm{D}^{\mathrm{b}}_c(X, \mathbb{C}) \to \mathrm{D}^{\mathrm{b}}_c(Y, \mathbb{C})$ is right t-exact concerning the perverse t-structure and the functor $f_!\colon \mathrm{D}^{\mathrm{b}}_c(X, \mathbb{C}) \to \mathrm{D}^{\mathrm{b}}_c(Y, \mathbb{C})$ is left t-exact concerning the perverse t-structure.*

In particular, combing with Theorem 1.2 we obtain

**Corollary 2.2.** *If $f$ is quasi-finite affine, then $f_*$ and $f_!$ are t-exact.*

Before delving into the proof, we will provide a few examples.

**Example 2.1.** *The open immersion $j\colon \mathbb{C}^\times \to \mathbb{C}$ is both affine and quasi-finite. We can show that $j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ and $j_!\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ are perverse by computing the stalks, since $\mathbb{D}j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1] \cong j_!\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$.*

| $j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ | $\{0\}$ | $\mathbb{C}^\times$ |
|---|---|---|
| $\mathrm{H}^0$ | $\underline{\mathbb{C}}$ | *0* |
| $\mathrm{H}^{-1}$ | $\underline{\mathbb{C}}$ | $\underline{\mathbb{C}}$ |

| $\mathbb{D}j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}[1]$ | $\{0\}$ | $\mathbb{C}^\times$ |
|---|---|---|
| $\mathrm{H}^0$ | *0* | *0* |
| $\mathrm{H}^{-1}$ | *0* | $\underline{\mathbb{C}}$ |

Now, we provide a counterexample.

**Example 2.2.** *The algebraic Hartogs lemma implies that $\mathcal{O}(\mathbb{C}^2 \setminus \{0\})$ is isomorphic to the $\mathcal{O}(\mathbb{C}^2)$. As a result, the embedding $j\colon \mathbb{C}^2 \setminus \{0\} \hookrightarrow \mathbb{C}^2$ is not an affine morphism. We can show that neither $j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]$ nor $j_!\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]$ is a perverse sheaf.*

*The motivation comes from the calculation of the cohomology of the stalk in $\{0\}$ of the sheaf $j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}$. By taking a open neighborhood of $\{0\}$ and open base change like case of $\mathbb{C}$ in talk 2, we will find $\mathrm{H}^k(j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}|_0)$ is isomorphic to $\mathrm{H}^k(\mathbb{S}^3, \mathbb{C})$. In particular,*

$$\mathrm{H}^3(j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}|_{\{0\}}) \cong \mathrm{H}^3(\mathbb{S}^3, \mathbb{C}) \cong \mathbb{C},$$

*Then* $\mathrm{H}^1(j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]|_{\{0\}})$ *is nontrivial, which is impossible for a perverse sheaf.*

*Since one can calculate* $j_!\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2] \cong \mathbb{D}j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]$,*, neither* $j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]$ *nor* $j_!\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]$ *is perverse.*

*We can also formally imply our arguments by perverse cohomology. By the open-closed triangle, we have the following two distinguished triangles:*

$$j_!\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2] \to \underline{\mathbb{C}}_{\mathbb{C}^2}[2] \to i_*\underline{\mathbb{C}}_{\{0\}}[2]; \qquad i_*\underline{\mathbb{C}}_{\{0\}}[-2] \to \underline{\mathbb{C}}[2] \to j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2],$$

*Notice* $i_*\underline{\mathbb{C}}_{\{0\}}$ *is perverse due to the property that* $i$ *is closed embedding, and* $\underline{\mathbb{C}}[2]$ *is perverse serving as shifted local system on* $\mathbb{C}^2$*. Upon application of the perverse cohomology* ${}^{\mathrm{p}}\mathrm{H}^*$*, the resulting short exact sequences in the Abelian category of perverse sheaf are as follows:*

$${}^{\mathrm{p}}\mathrm{H}^{-2}(\underline{\mathbb{C}}[2]) \cong 0 \to {}^{\mathrm{p}}\mathrm{H}^{-2}(i_*\underline{\mathbb{C}}_{\{0\}}[2]) \cong i_*\underline{\mathbb{C}}_{\{0\}} \to {}^{\mathrm{p}}\mathrm{H}^{-1}(j_!\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]);$$

$${}^{\mathrm{p}}\mathrm{H}^{2}(j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]) \to {}^{\mathrm{p}}\mathrm{H}^{2}(i_*\underline{\mathbb{C}}_{\{0\}}[-2]) \cong i_*\underline{\mathbb{C}}_{\{0\}} \to {}^{\mathrm{p}}\mathrm{H}^{2}(\underline{\mathbb{C}}[2]) \cong 0,$$

*In particular, neither* $j_*\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]$ *nor* $j_!\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}[2]$ *is perverse, since they exhibit nontrivial perverse cohomology in nonzero degrees.*

If we assume the following criterion, the main theorem2.1 is immediate from the property of affine morphism.

**Lemma 2.3** ([Ach21] Theorem 3.5.3)**.** *Let* $X$ *be a variety and let* $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_c(X)$*. Then, the following are equivalent.*

1. *(1)* $\mathcal{F} \in {}^{\mathrm{p}}\mathrm{D}^{\leq 0}_c(X)$*.*
2. *(2) For any affine open subvariety* $U \subseteq X$*, we have* $R\Gamma(U, \mathcal{F}|_U) \in \mathrm{D}^b(\mathbb{C} - \mathbf{Mod}^{fg})^{\leq 0}$*.*

Finally, we can prove the theorem 2.1.

*Proof of Theorem 2.1.* In the context of a coefficient ring of constructible sheaves being $\mathbb{C}$, we can establish that $f_! \cong \mathbb{D}f_*\mathbb{D}$ holds true. From this, we can infer that the left t-exactness of $f_!$ is equivalent to the right t-exactness of $f_*$ by Lemma 1.6. Let's consider $\mathcal{F} \in {}^{\mathrm{p}}\mathrm{D}^{\leq 0}_c(X)$. According to the lemma 2.3, it suffices to prove that for any affine open subvariety $U \subseteq Y$, $R\Gamma(U, f_*\mathcal{F}|_U) \in \mathrm{D}^b(\mathbb{C} - \mathbf{Mod}^{fg})^{\leq 0}$.

We know that $f^{-1}(U)$ is also affine because $f$ is affine. Using the open / smooth base change, we obtain $R\Gamma(U, f_*\mathcal{F}|_U) \cong R\Gamma(f^{-1}(U), \mathcal{F})$. By lemma 2.3, it follows that $R\Gamma(f^{-1}(U), \mathcal{F}) \in \mathrm{D}^b(\mathbb{C} - \mathbf{Mod}^{fg})^{\leq 0}$. Consequently, we can declare that for $U \subseteq Y$, $R\Gamma(U, f_*\mathcal{F}|_U) \in \mathrm{D}^b(\mathbb{C} - \mathbf{Mod}^{fg})^{\leq 0}$. □

We will start the proof of this criterion after some statements on the cohomology bound. To begin with, we will demonstrate the existence of an affine open $U \subseteq X$ such that $\mathbf{H}^{\dim \operatorname{supp} \mathcal{F}}(U, \mathcal{F}|_U) \neq 0$. We will use this to establish a criterion for a constructible sheaf in ${}^{\mathrm{p}}\mathrm{D}^{\leq 0}_c(X)$. Finally, we can prove the main theorem 2.1.

We will now present some results on the cohomology bound.

**Theorem 2.4** ([Ach21] Theorem 2.74, 2.75)**.** *If* $\mathcal{F}$ *is constructible, then* $\mathbf{H}^i(X, \mathcal{F})$ *and* $\mathbf{H}^i_c(X, \mathcal{F})$ *vanish for* $i > 2\dim X$*.*

From this we can deduce a result concerning the cohomology bound of $f_*$ and $f_!$.

**Corollary 2.5.** *Let* $f : X \to Y$ *be a morphism of varieties, and* $\mathcal{F}$ *is constructible. Then cohomology sheaves* $\mathrm{H}^i(f_*\mathcal{F})$ *and* $\mathrm{H}^i(f_!\mathcal{F})$ *vanish for* $i > 2\dim X$*.*

In addition, Artin's vanishing theorem addresses the cohomological bound on an affine variety X in the following way:

**Theorem 2.6.** *([Ach21] Theorem 2.6.2)*
*If* $X$ *is affine and* $\mathcal{F}$ *is constructible, then* $\mathbf{H}^i(X, \mathcal{F})$ *and* $\mathbf{H}^i_c(X, \mathcal{F})$ *are* $0$ *for* $i > \dim \operatorname{supp}(\mathcal{F})$*. Specifically,* $\mathbf{H}^i(X, \mathcal{F})$ *and* $\mathbf{H}^i_c(X, \mathcal{F})$ *are* $0$ *for* $i > \dim X$*.*

Then, we outline a non-vanishing theorem on an open affine variety.

**Proposition 2.1** ([Ach21] Theorem 3.5.1)**.** *If the dimension of the support of the constructible sheaf $\mathcal{F}$ is denoted by $n$, then there exists an affine open subvariety $U \subseteq X$ such that the $n$-th cohomology group $\mathbf{H}^n(U, \mathcal{F}|_U)$ is nonzero.*

*Sketch.* Without loss of generality, we assume that $X$ is an affine variety. Let $\dim X = m$. It is evident that $m \geq n = \dim \operatorname{supp} \mathcal{F}$. Now, we proceed by induction on $n$.

When $n = 0$: we have $\mathbf{H}^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F}) \cong \bigoplus_{x \in \operatorname{supp} \mathcal{F}} \mathcal{F}_x \neq 0$. When $n > 0$, we will proceed with induction on $m$.

Step 1 To reduce the case to $X = \mathbb{A}^m$, we use the Noether normalization theorem to establish a finite morphism $f \colon X \to \mathbb{A}^m$. Subsequently, $f_*\mathcal{F}$ is constructible, and $\operatorname{supp}(f_*\mathcal{F}) = f(\operatorname{supp} \mathcal{F})$. In particular, through open base change, if we take any open affine $U' \subseteq \mathbb{A}^m$ and let $U = f^{-1}(U')$, the isomorphism $\mathbf{H}^n(U, \mathcal{F}|_U) \cong \mathbf{H}^n(U', (f_*\mathcal{F})|_{U'})$ is notable. Consequently, demonstrating the result for $X = \mathbb{A}^m$ suffices.

Step 2 Reduction to the case $m = n$. Select a sufficiently small open affine subvariety $U$ such that the following conditions hold:

(1) $\mathcal{F}|_U$ forms a local system

(2) $Z \coloneqq X \setminus U$ is affine closed subvariety such that the restriction of projection $p \colon \mathbb{A}^m \to \mathbb{A}^{m-1}$ to $Z$ is a finite surjection of degree $\geq 2$.

Now, if $m > n$, then $\mathcal{F}|_U$ must be zero and $\operatorname{supp} \mathcal{F} \subseteq Z$, Therefore, by step 1, we can reduce the proof to the case $m = n$.

Step 3 Select a non-empty irreducible affine smooth open subvariety $V_0 \subseteq \mathbb{A}^{m-1}$ through [Ach21] Lemma 2.5.4 such that the following properties hold.

(a) $Z_0 \coloneqq V_0 \times \mathbb{A}^1 \bigcap Z$ is a simple normal crossing divisor in $V_0 \times \mathbb{A}^1$, and it is finite surjective étale over $V_0$.

(b) the map $pr_1 \colon V_0 \times \mathbb{A}^1 \to V_0$ is a transverse locally trivial fibration with respect to $Z_0$.

Denote $U_0 \coloneqq p^{-1}(V_0) \bigcap U$ ,which is affine as the intersection of two affine varieties. In particular, by property (b), the pullback of projection $p \colon U \to \mathbb{A}^{m-1}$ along $V_0$, which we denote as $q \colon U_0 = p^{-1}(V_0) \bigcap U \to V_0$, is locally trivial fibration. It can be denoted as a pullback diagram.

$$\begin{tikzcd} U_0 \arrow[r, hook] \arrow[d, "q"] & U \arrow[d, "p"] \\ V_0 \arrow[r, hook] & \mathbb{A}^{m-1} \end{tikzcd}$$

Hence, through [Ach21] 2.4.5, $q_*(\mathcal{F}|_{U_0}) \in \mathrm{D}^b_{locf}(V_0, \mathbb{C})$ and for any $y \in V_0$, $q_*(\mathcal{F}|_{U_0})_y \cong R\Gamma(\mathcal{F}|q^{-1}(y))$.

According to property (1) in step 2 and the above property of locally trivial fibration, $\mathcal{F}' \coloneqq \mathrm{H}^1(q_*(\mathcal{F}|_{U_0}))$ is a local system on $V_0$. Denote $q^{-1}(y) = p^{-1}(y) \bigcap U_0$. Using property (2) in step 2 and property (a) in Step 3, $q^{-1}(y)$ is a complement of at least two points in $\mathbb{A}^1$. By the calculation in [Ach21] Lemma B 3.6 in $\mathbb{A}^1$, $\mathrm{H}^1(q^{-1}(y), \mathcal{F}|q^{-1}(y)) \neq 0$, which shows that $\mathcal{F}' \neq 0$.

Upon explicit calculation on [Ach21] lemma B.3.6 or Artin's vanishing theorem 2.6, we can deduce $\mathrm{H}^k(q_*(\mathcal{F}|_{U_0}))$ vanishes for $k \geq 2$. Consequently the trunction-distinguished triangle to $q_*(\mathcal{F}|_{U_0})$ reduces to

$$\mathrm{H}^0(q_*(\mathcal{F}|_{U_0})) \to q_*(\mathcal{F}|_{U_0}) \to \mathcal{F}'[-1],$$

Subsequency, we can use the induction hypothesis to $\mathcal{F}'$ and get an affine open subvariety $V_1 \subseteq V_0 \subset \mathbb{A}^{m-1}$ such that $\mathbf{H}^{m-1}(V_1, \mathcal{F}'|_{V_1}) \neq 0$. Let $U_1 := q^{-1}(V_1) \bigcap U_0$, which is affine as an intersection of two affine varieties. It can be viewed as the pullback diagram:

$$\begin{tikzcd}
U_1 \arrow[r, hook] \arrow[d, "r"] & U_0 \arrow[r, hook] \arrow[d, "q"] & U \arrow[d, "p"] \\
V_1 \arrow[r, hook] & V_0 \arrow[r, hook] & \mathbb{A}^{m-1}
\end{tikzcd}$$

After restricting the truncation-distinguished triangle to $V_1$, we obtain the following exact sequence:

$$0 \cong \mathbf{H}^m(V_1, \mathrm{H}^0(q_*(\mathcal{F}|_{U_0}))) \to \mathbf{H}^m(V_1, q_*(\mathcal{F}|_{U_0})) \to \mathbf{H}^{m-1}(V_1, \mathcal{F}') \to 0 \cong \mathbf{H}^{m+1}(V_1, \mathrm{H}^0(q_*(\mathcal{F}|_{U_0}))),$$

Here $\mathbf{H}^m(V_1, \mathrm{H}^0(q_*(\mathcal{F}|_{U_0}))) \cong 0$ and $\mathbf{H}^{m+1}(V_1, \mathrm{H}^0(q_*(\mathcal{F}|_{U_0}))) \cong 0$ since we apply Artin's vanishing theorem to the constructible sheaf $\mathrm{H}^0(q_*(\mathcal{F}|_{U_0}))$ on $m-1$ diminsional open affine subvariety $V_1$.

As a result, using open base change, $\mathbf{H}^m(U_1, \mathcal{F}|_{U_1}) \cong \mathbf{H}^m(V_1, q_*(\mathcal{F}|_{U_0})) \cong \mathbf{H}^{m-1}(V_1, \mathcal{F}') \neq 0$.

□

Following this proposition, we can derive a lemma that helps validate perversity.

**Lemma 2.7** ([Ach21]). *Theorem 3.5.2) Let $\mathcal{F} \in {}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(X)$. For any affine open subvariety $U \subset X$, we have the complex of finite-dimensional vector space $R\Gamma(U, \mathcal{F}|_U) \in \mathrm{D}^b(\mathbb{C} - \mathbf{Mod}^{fg})^{\leq 0}$. Moreover, if $\mathcal{F} \notin {}^{\mathrm{p}}\mathrm{D}_c^{\leq -1}(X)$, then there exists an affine open subvariety $U \subset X$ such that $\mathbf{H}^0(U, \mathcal{F}|_U) \neq 0$.*

*Proof.* To prove the first half argument when U=X, we can utilize a spectral sequence that converges on the E2 page, given by the following:

$$E_2^{pq} = \mathbf{H}^q(X, \mathrm{H}^p(\mathcal{F})) => \mathbf{H}^{p+q}(X, \mathcal{F}),$$

Since $\mathcal{F}$ is an element of ${}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(X)$, it follows that $\dim \operatorname{supp} \mathrm{H}^p(\mathcal{F}) \leq -p$. According to Artin's vanishing, i.e. Theorem 2.6, $\mathbf{H}^q(X, \mathrm{H}^p(\mathcal{F})) = 0$ for $q > -p$, which means $p + q > 0$. Thus, due to the convergence of the spectral sequence, we can conclude $\mathbf{H}^{p+q}(X, F) = 0$ if $p + q > 0$.

In the context of the second statement, we are looking at a truncation-distinguished triangle for a perverse t-structure, which means the sequence

$${}^{\mathrm{p}}\tau^{\leq -1}\mathcal{F} \to \mathcal{F} \to {}^{\mathrm{p}}\tau^{\geq 0}\mathcal{F},$$

From this, we observe that ${}^{\mathrm{p}}\tau^{\geq 0}\mathcal{F}$ is a nonzero perverse sheaf, based on the assumption that $\mathcal{F}$ does not belong to ${}^{\mathrm{p}}\mathrm{D}_c^{\leq -1}(X)$ and that $\mathcal{F} \in {}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(X)$.

In the case where there is an affine open subvariety $U \subset X$ such that $\mathbf{H}^0(U, {}^{\mathrm{p}}\tau^{\geq 0}\mathcal{F}|_U) \neq 0$, we can conclude that $\mathbf{H}^0(U, \mathcal{F}|_U) \neq 0$ since $\mathbf{H}^1(U, {}^{\mathrm{p}}\tau^{\leq -1}\mathcal{F}|_U) = 0$ based on the first part of the argument. This discussion simplifies the assertion to the case where $F$ is perverse. Additionally, every perverse sheaf has a simple perverse quotient, allowing us to simplify to $\mathcal{F} = IC(V, \mathcal{L})$ simple perverse, where $V$ is smooth locally closed and $\mathcal{L}$ is a finite type local system. Specifically, the support of $IC(V, \mathcal{L})$ is $\overline{V}$. We can further simplify the proof by replacing $X$ with $\overline{V}$, assuming that $V$ is open dense in X and affine. We can find the desired $U$ using the proposition 2.1.

□

Finally, we come to the proof of the criterion on the upper boundedness of the constructible sheaf.

*Proof of Lemma 2.3.* The first argument implies the second by Lemma 2.7.

Let us prove the other direction. Suppose $\mathcal{F} \notin {}^{\mathrm{p}}\mathrm{D}_c^{\leq 0}(X)$. Since the perverse t-structure is bounded, take the smallest positive integer $n$ such that $\mathcal{F} \in {}^{\mathrm{p}}\mathrm{D}_c^{\leq n}(X)$ and $\mathcal{F} \notin {}^{\mathrm{p}}\mathrm{D}_c^{\leq n-1}(X)$. Consequently, it satisfies the condition of the second argument in Lemma 2.7. It implies that there exists an affine open subvariety $U \subseteq X$ such that $\mathbf{H}^n(U, \mathcal{F}|_U) \neq 0$, which contradicts to the assumption that $R\Gamma(U, \mathcal{F}|_U) \in \mathrm{D}^b(\mathbb{C} - \mathbf{Mod}^{fg})^{\leq 0}$. $\square$

# 3 Smooth Descent

Perverse sheaves satisfy the descent for the smooth topology ([Ach21] Section 3.7), which means that they can be glued together from perverse sheaves on an open cover, which is another way that perverse sheaves behave like sheaves. In particular, a morphism of perverse sheaf can be glued from an open cover. This behavior of the perverse sheaf differs from that of the constructible sheaf, which does not satisfy the smooth descent. I will illustrate this by showing a counterexample.

**Example 3.1.** *Considering the constant sheaf $\underline{\mathbb{C}}_{\mathbb{P}^1}$ on $\mathbb{P}^1$. By the equivalent definition of sheaf cohomology,*

$$\mathbf{H}^2(\mathbb{P}^1, \underline{\mathbb{C}}_{\mathbb{P}^1}) \cong \mathbf{Hom}_{\mathrm{D}_c^{\mathrm{b}}(\mathbb{P}^1)}(\underline{\mathbb{C}}_{\mathbb{P}^1}, \underline{\mathbb{C}}_{\mathbb{P}^1}[2]),$$

*In particular, there is a nonzero morphism $\alpha \in \mathbf{Hom}_{\mathrm{D}_c^{\mathrm{b}}(\mathbb{P}^1)}(\underline{\mathbb{C}}_{\mathbb{P}^1}, \underline{\mathbb{C}}_{\mathbb{P}^1}[2])$ since $\mathbf{H}^2(\mathbb{P}^1, \underline{\mathbb{C}}_{\mathbb{P}^1}) \cong \mathbb{C}$. However if we restrict $\alpha$ to any subvariety $\mathbb{A}^1$, it is trivial in $\mathbf{Mor}(\mathrm{D}_c^{\mathrm{b}}(\mathbb{A}^1))$ since $\mathbf{H}^2(\mathbb{A}^1, \underline{\mathbb{C}}_{\mathbb{A}^1}) \cong 0$ from the contractility of $\mathbb{A}^1$ under complex analytic topology. As $\mathbb{P}^1$ admits an open an affine cover given by subvarieties isomorphic to $\mathbb{A}^1$, $\alpha$ cannot be obtained by gluing on this open cover.*

For an exact meaning, we first need to recall the descent data.

**Definition 3.1.** *Let $f\colon X \to Y$ be a smooth surjective morphism of relative dimension $d$, and let $\mathcal{F} \in$ $\mathbf{Perv}(X)$. Recall that we denote $f^\dagger := f^*[d]$. A descent data for $\mathcal{F}$ with respect to $f$ is an isomorphism*

$$\phi\colon \mathrm{pr}_1^\dagger \mathcal{F} \cong \mathrm{pr}_2^\dagger \mathcal{F} \quad \text{in} \quad \mathbf{Perv}(X \times_Y X)$$

*such that the following diagram in $\mathbf{Perv}(X \times_Y X \times_Y X)$ follows*

$$\begin{tikzcd}
\mathrm{pr}_1^\dagger \mathcal{F} \arrow[rr, "\mathrm{pr}_{13}^\dagger \phi"] \arrow[dr, "\mathrm{pr}_{12}^\dagger \phi"'] & & \mathrm{pr}_3^\dagger \mathcal{F} \\
& \mathrm{pr}_2^\dagger \mathcal{F} \arrow[ur, "\mathrm{pr}_{23}^\dagger \phi"'] &
\end{tikzcd}$$

Then, an essential interpretation that perverse sheaves satisfy descent for the smooth topology is the following property.

**Theorem 3.1.** *(Part of [Ach21] Theorem 3.7.4) Given any descent data for $\mathcal{F}$ concerning smooth surjective $f$, there is a $\mathcal{G} \in \mathbf{Perv}(Y)$ such that $\mathcal{F} \cong f^\dagger \mathcal{G}$.*

**Remark 3.1.** *We can upgrade the descent data to a category $\mathbf{Desc}(X, f)$ by defining the morphism between the descent datum. A morphism between descent datum $h\colon (\mathcal{F}_1, \phi_1) \to (\mathcal{F}_2, \phi_2)$ is a morphism of perverse sheaves $h\colon \mathcal{F}_1 \to \mathcal{F}_2$ such that the diagram*

$$\begin{tikzcd}
\mathrm{pr}_1^\dagger \mathcal{F}_1 \arrow[r, "\phi_1"] \arrow[d, "\mathrm{pr}_1^\dagger \phi"] & \mathrm{pr}_2^\dagger \mathcal{F}_1 \arrow[d, "\mathrm{pr}_2^\dagger \phi"] \\
\mathrm{pr}_1^\dagger \mathcal{F}_2 \arrow[r, "\phi_2"] & \mathrm{pr}_2^\dagger \mathcal{F}_2
\end{tikzcd}$$

*Taking $\mathcal{G} \in \mathbf{Perv}(Y)$, there is canonical descent data for $f^\dagger \mathcal{G}$ given by the isomorphism $f \circ pr_1 = f \circ pr_2$. In fact, $pr_1^*(f^*(\mathcal{G})) \cong (f \circ pr_1)^*(\mathcal{G}) = (f \circ pr_2)^*(\mathcal{G}) \cong pr_2^*(f^*(\mathcal{G}))$.*

*All above makes sure we have a functor*

$$f^\dagger \colon \mathbf{Perv}(Y) \to \mathbf{Desc}(X, f),$$

*In addition, this functor induces an equivalence of category by [Ach21] Theorem 3.7.4.*

# References

[Ach21] Pramod N. Achar. "Perverse Sheaves and Applications to Representation Theory". In: *Mathematical surveys and monographs,American Mathematical Society*, 258 (2021).