Computational Methods for Perverse Sheaves

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Abstract

These are notes prepared for the seminar on perverse sheaves organized by Dr. Dawydiak during the SS 2024. The goal of this talk is to see perverse and constructible sheaves in action with a focus on curves, especially on intersection homology and how it relates to Poincaré duality. In particular, we will not prove any new results today but we will see how $\mathbb{G}_m$-actions and small morphisms are used in practice.

Contents

1 Perverse Sheaves on a Disk

We follow [Wil] in this section.
The goal of this first section is to describe a simpler description of the category of perverse sheaves in a simple case. Remember that we call a sheaf of complex vector spaces $\mathcal{F}$ *constructible* on X a variety if there exists a stratification $\coprod X_s$ of X such that $\mathcal{F}|_{X_s}$ is a local system.
How does this look if $X = D$ is the closed unit disk? Let $U = D \setminus 0 \subset X$ and let us suppose that $\mathcal{F}$ is constructible with respect to the stratification $\{0\} \subset D$, we let $i\colon \{0\} \hookrightarrow D \hookleftarrow U\colon j$ the complementary immersions. Then $\mathcal{F}|_U = \mathcal{L}$ is a local system

and $\mathcal{F}|_0 = M$ is a vector space.
How do M and $\mathcal{L}$ interact? There is a map $\alpha\colon M = H^0(\{0\}, i^*\mathcal{F}) \to H^0(\{0\}, i^*j_*j^*\mathcal{F}) = \mathcal{L}^{\pi_1}$ obtained from pulling back the natural map $\mathcal{F} \to j_*j^*\mathcal{F}$ and it turns out that the triple $(\mathcal{L}, M, \alpha)$ uniquely determines the constructible sheaf $\mathcal{F}$. The map α precisely gives the transition maps from a simply connected neighborhood of 0 to the punctured neighborhood.
Hence, constructible sheaves can be described quite easily in terms of linear algebra data, is there a similar description for perverse sheaves on curves?

We denote Perv_0 the category of perverse sheaves on D which are constructible with respect to $\{0\} \subset D$. The following is a special case of a theorem originally due to Verdier, see [Ver83].

Theorem 1 Perv_0 *is equivalent to the category of pairs* $(V, V_0, \mu, \alpha, \beta)$ *with* V, V_0 *vector spaces,* $\mu \in GL(V)$ *and* α, β *maps making*

$$\begin{array}{ccc} V & \xrightarrow{\mu-\mathrm{Id}} & V \\ & {}_{\alpha}\searrow \qquad \nearrow_{\beta} & \\ & V_0 & \end{array}$$

commute.

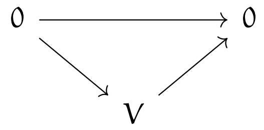
We will not prove this but we will describe the involved functor and give concrete descriptions of V and V_0 in known cases.
Let $\mathcal{F} \in \mathrm{Perv}_0$, at the end, the associated V will be the vector space associated to the local system on $U \setminus \{0\}$ and μ the monodromy action.
The key question one has to answer is how to take "stalks at 0" of a perverse sheaf, this will be the vector space V_0.

Let $\nu\colon D_{\mathrm{Re}<\gamma} \hookrightarrow D$ the inclusion, the triangle associated to this open inclusion and it's complementary closed subset gives rise to a long exact sequence

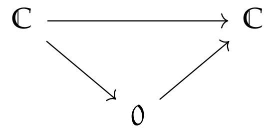
$$\cdots \to H^{-1}(D, \mathcal{F}_{D_{\mathrm{Re}>\gamma}}) \to H^0(D, \nu_!\nu^!\mathcal{F}) \to H^0(D, \mathcal{F}) \to H^0(D, \mathcal{F}_{D_{\mathrm{Re}\geq\gamma}})$$

and in this sense, the complex $\nu_!\nu^!\mathcal{F}$ measures the difference of the sections of the sheaf $\mathcal{F}$ as γ varies from positive to negative.
It turns out that $\nu_!\nu^!\mathcal{F}$ is always concentrated in degree 0 as one can easily prove, see [Wil, prop. 8.2]. We set $V_0 = H^0(\nu_!\nu^!\mathcal{F})$.
We omit the description of the maps α and β, see [Wil, section 8.1] and [Ach21, section 4.3] for a more detailed discussion.

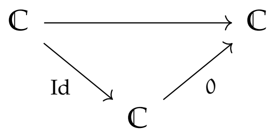
Example 1 *1. For instance, let* $\mathcal{F} = i_*\underline{\mathbb{C}}_{\{0\}}$*, then clearly* $V = 0$ *and from the above long exact sequence one sees that* $V_0 = \mathbb{C}$ *so the associated triangle is*



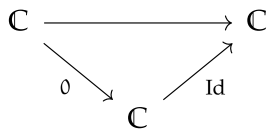
2. *If* $\mathcal{F} = j_*\underline{\mathbb{C}}_U[1]$*, then by similar arguments one sees that*



3. *If* $\mathcal{F} = j_!\underline{\mathbb{C}}_U[1]$*, then, if one traces through the construction of the maps above, we obtain*



4. *If* $\mathcal{F} = j_*\underline{\mathbb{C}}_U[1]$*, then one can see that*



2 Reminders on intersection cohomology

Fix a locally closed subvariety $h\colon Y \hookrightarrow X$ and a finite type local system $\mathcal{L}$ on Y.
Given $h\colon Y \to X$ a locally closed embedding, we defined the *intermediate extension* functor

$$\begin{aligned} h_{!*}\colon \mathrm{Perv}(Y) &\to \mathrm{Perv}(X) \\ \mathcal{F} &\mapsto \mathrm{im}({}^pH^0(h_!\mathcal{F}) \to {}^pH^0(h_*\mathcal{F})). \end{aligned}$$

In particular, we defined the intersection cohomology complex of a pair $(Y \subset X, \mathcal{L})$ $\mathrm{IC}(Y, \mathcal{L}) = h_{!*}(\mathcal{L}[\dim Y])$. The most relevant case of this construction was the intersection cohomology of X itself which we defined as

$$\mathrm{IC}(X) \coloneqq \mathrm{IC}(X_{sm}, \underline{\mathbb{C}}_{X_{sm}}).$$

When X is smooth, then the intersection cohomology complex is the trivial local system and [Ach21, lemma 3.3.13] precisely gives Poincaré duality:

$$\mathbb{D}(IC(X)) = IC(X, \mathbb{C}^\vee).$$

In fact, this Poincaré duality holds in a slightly more general context, namely for what are called *rationally smooth* varieties.

Definition 1 (Rationally Smooth Variety) *An irreducible variety X is called rationally smooth if one of the following equivalent conditions is satisfied*

1. $\omega_X = \underline{\mathbb{C}}_X[2\dim X]$
2. $IC(X) = \underline{\mathbb{C}}_X[\dim X]$
3. *For all $x \in X$ we have*

$$H^i(IC(X)_x) = \begin{cases} 1 \text{ if } i = -\dim X \\ 0 \text{ else} \end{cases}$$

We prove that these characterizations are in fact equivalent

Proof ([Ach21, Ex. 3.10.3]) Throughout let $j\colon X_{sm} \to X$ be the open embedding and $i\colon Y \coloneqq X \setminus X_{sm} \to X$ the complementary closed embedding.
Suppose 1 holds true, to show 2 holds true it suffices by [Ach21, lemma 3.3.4] to show that $\underline{\mathbb{C}}[\dim X]$ satisfies the two following properties

$$\text{The support of } \underline{\mathbb{C}}_X[\dim X] \text{ is } X, \underline{\mathbb{C}}_X[\dim X]|_{X_{sm}} = \mathbb{C}[\dim X_{sm}] \text{ and}$$

$$i^!\underline{\mathbb{C}}_X[\dim X] \in {}^pD^b_c(Y)^{\geq 1} \text{ and } i^*\underline{\mathbb{C}}_X[\dim X] \in {}^pD^b_c(Y)^{\leq -1}$$

The first two properties are obvious since $\dim X = \dim X_{sm}$ by generic smoothness.
To prove the two last properties, first note that $\mathbb{D}(i^*\underline{\mathbb{C}}_X[\dim X]) = i^!\mathbb{D}(\underline{\mathbb{C}}_X[\dim X]) = i^!\underline{\mathbb{C}}_X[\dim X]$ and thus it suffices to prove that $i^*\underline{\mathbb{C}}_X[\dim X] \in {}^pD^b_c(Y)^{\leq -1}$ or equivalently, that $i^*\underline{\mathbb{C}}_X[\dim X][-1] = \underline{\mathbb{C}}_Y[\dim X - 1]$ is a coconnective object in the perverse t-structure on $D^b_c(Y)$.
However, X^{sm} is dense in X so it's complement has positive codimension and this is automatically satisfied.
The implication $2 \implies 3$ is obvious and, since there always is a canonical non-zero map $\underline{\mathbb{C}}[\dim X] \to IC(X)$, the converse direction is too.
Finally, suppose 2 holds, then

$$\mathbb{D}(IC(X)) = IC(X) \iff RHom(\underline{\mathbb{C}}_X, \omega_X[-\dim X]) = \underline{\mathbb{C}}_X[\dim X] \iff \omega_X = \underline{\mathbb{C}}_X[2\dim X].$$

□

As the linguistics suggest, every smooth variety is rationally smooth but there do exist non-smooth varieties which are rationally smooth, see [BL00, theorem 10.1.1, 2].

We will see examples of varieties which are not rationally smooth once we have more computational techniques.

3 Semismall morphisms

It is easy to see that pushforward along a finite morphism is t-exact for the perverse t-structure, one might ask for less stringent conditions that still guarantee t-exactness. Perhaps the easiest example of this phenomenon is the following, notice that the perverse t-structure on a point $\operatorname{Spec}\mathbb{C}$ coincides with the canonical one, hence the projection map $p\colon \mathbb{P}^1 \to \operatorname{Spec}\mathbb{C}$ has no chance of being t-exact.
So are there conditions on a morphism of varieties that ensure it is t-exact? While such a general result is difficult to achieve (though possible if one restricts to nice stratifications on X, see [Ach21, theorem 3.8.9]), one could at least hope for the pushforwards of local systems to be perverse. This turns out to be the case under assumptions on the dimension of fibers.

Definition 2 (Semismall morphisms) *Let X be an irreducible variety. A morphism $f\colon X \to Y$ is called*

- ***semismall*** *if Y admits a stratification $\{Y_t\}_{t\in\mathcal{T}}$ such that for all $t \in \mathcal{T}$ and each point $y \in Y_t \cap f(X)$ we have*

$$\dim f^{-1}(y) \leq \frac{1}{2}(\dim X - \dim Y_t).$$

- ***small*** *with respect to $W \subset Y$ an open dense subset if Y admits a stratification $\{Y_t\}_{t\in\mathcal{T}}$ such that W is a union of strata, for all $y \in W$ the fiber $f^{-1}(y)$ is finite and for all $y \in Y_t \cap f(X)$*

$$\dim f^{-1}(y) < \frac{1}{2}(\dim X - \dim Y_t)$$

Notice that neither of these conditions are satisfied for $\mathbb{P}^1 \to \operatorname{Spec}\mathbb{C}$.
We will not prove the following theorems in today's talk.

Theorem 2 ([Ach21, theorem 3.8.4]) *Let $f\colon X \to Y$ be a proper semismall morphism with X smooth and connected and let $\mathcal{L}$ be a local system on X, then $f_*\mathcal{L}[\dim X]$ is perverse on Y.*

The next question one might ask is whether one can give an explicit description of the perverse sheaf appearing in this theorem.

Theorem 3 ([Ach21, prop. 3.8.7]) *Let* X *be a smooth connected variety, and let* $f\colon X \to Y$ *be a proper, small morphism with respect to* $W \subset Y$*. Let* $f' := f|_{f^{-1}(W}\colon f^{-1}(W) \to W$ *and* $h\colon W \hookrightarrow Y$ *be the inclusion map. Then, for any finite type local system* $\mathcal{L}$ *on* X *we have*

$$f_*\mathcal{L}[\dim X] = h_{!*}(f'_*\mathcal{L}|_{f^{-1}(W)}[\dim X]).$$

Proof Let $i\colon \bar{W} \setminus W \hookrightarrow Y$ be the inclusion. We will need the two following facts that we will not prove

- if $\mathcal{F} \in {}^{p}D^{\flat}_{locf}(X)^{\leq 0}$ then $i^*(f_!\mathcal{F}) \in {}^{p}D^{\flat}_{locf}(\bar{W} \setminus W)^{\leq -1}$
- if $\mathcal{F} \in {}^{p}D^{\flat}_{locf}(X)^{\geq 0}$ then $i^!(f_*\mathcal{F}) \in {}^{p}D^{\flat}_{locf}(\bar{W} \setminus W)^{\geq 1}$

From the characterization of intermediate extensions we have seen, the theorem now follows easily. □

This result is useful in computations thanks to the following lemma. The proof is an easy exercise, one uses that the normalization map is finite so in particular small.

Lemma 4 ([HTT08, prop. 8.2.31]) *Let* X *be a projective variety and* $\pi\colon \tilde{X} \to X$ *it's normalization, then there is an isomorphism*

$$R\pi_* IC_{\tilde{X}} = IC_X$$

In particular, $IH(X) = IH(\tilde{X})$*.*

Notice that normalizations typically don't induce isomorphisms on singular (co)homology as the following example shows.

Example 2 ([HTT08, Example 8.2.32]) *Let* C *be the cuspidal cubic, recall that it's normalization is* $\tilde{C} = \mathbb{P}^1 = S^2$*. Now,* C *is homeomorphic to a pinched torus, whose homology is*

$$H_0(C) = H_1(C) = H_2(C) = \mathbb{C}.$$

So $H_*(C) \neq H_*(\mathbb{P}^1)$*. However, the above lemma implies that*

$$IH^\bullet(C) = IH^\bullet(\mathbb{P}^1)$$

So singular homology and intersection homology disagree. In particular, C *is not rationally smooth.*

4 $\mathbb{G}_m$-actions and a Longer Example

We now turn to a longer example in which we will compute, among other things, the intersection cohomology sheaves of a singular variety. One key computational aspect will be to notice that finding (semi-)small resolutions can be a useful asset in computations.

For this, we will need to understand $\mathbb{G}_m$-actions on complex varieties, so throughout let $\sigma\colon \mathbb{G}_m \times X \to X$ be an action.

Throughout, we fix the standard embedding $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ which misses the origin.

Definition 3 (Attracting Action) *If for every $t \in X$, the orbit map extends to a map $\tilde{\sigma}_t\colon \mathbb{A}_1 \to X$, we call σ attracting.*

If the action is attracting, the image $\tilde{\sigma}_t(0)$ will always be a fixed point of the action and the association $t \mapsto \tilde{\sigma}_t(0)$ assembles to a well-defined map of varieties $p\colon X \mapsto X^{\mathbb{G}_m}$.

Definition 4 (Weakly Equivariant) *A complex $\mathcal{F} \in D^b_c(X)$ is weakly $\mathbb{G}_m$-equivariant if $\sigma^*\mathcal{F} \simeq \underline{\mathbb{C}}_{\mathbb{G}_m} \boxtimes \mathcal{F}$.*

Theorem 5 (Homotopy of sheaves) *Let σ be an attracting action on X, $i\colon X^{\mathbb{G}_m} \hookrightarrow X$ the natural inclusion, $p\colon X \mapsto X^{\mathbb{G}_m}$ the projection and $\mathcal{F}$ a weakly equivariant complex on X, then there is a natural isomorphism*

$$p_*\mathcal{F} \to i^*\mathcal{F}.$$

We will apply this in the following long example. The final goal will be to explicitly determine the intersection cohomology on a given variety, along the way we will see different techniques one can use to compute all kinds of relevant sheaves.

Example 3 *Let $X = \{x \in \mathrm{Mat}_{2\times 2} | \det x = 0\}$ and $\tilde{X} = \left\{(x, L) \in X \times \mathbb{P}^1 | L \in \ker x\right\}$. There is an obvious map $q\colon \tilde{X} \to X$, one easily sees that the fiber over 0 is $\mathbb{P}^1$ while q is an isomorphism away from 0.*

One easily checks that $\tilde{X}$ is smooth by writing out charts and it is also easy to see that $\tilde{X} \to X$ is a semismall morphism. We now compute pushforwards of various sheaves

$q_*\underline{\mathbb{C}}_{\tilde{X}}$

We use cohomology and base change to compute the fiber over 0, recalling that $\mathbb{P}^1 \simeq S^2$ we find

-	*U*	$\{0\}$
2	*0*	$\mathbb{C}$
1	*0*	*0*
0	$\mathbb{C}$	$\mathbb{C}$

$j_*\underline{\mathbb{C}}_U$

I will present here two different computations that lead to the correct answer, the first method was not presented during the seminar.

Spectral Sequence Method

Notice that there is a natural $\mathbb{G}_m$-action on X by scalar multiplication which extends to a $\mathbb{G}_m$-action on $\tilde{X}$ whose fixed points are precisely $q^{-1}(0)$.
Let $\tilde{j}\colon p^{-1}(U) \to \tilde{X}$ be the natural inclusion, $\tilde{i}\colon \mathbb{P}^1 = \tilde{X}^{\mathbb{G}_m} \hookrightarrow \tilde{X}$ and $p\colon \tilde{X} \to \tilde{X}^{\mathbb{G}_m}$ the attraction map from above. These fit into the following diagram

$$\begin{array}{ccccc} \tilde{X}^{\mathbb{G}_m} & \xrightarrow[\tilde{i}]{} & \tilde{X} & \xleftarrow{\tilde{j}} & q^{-1}(U) \\ \downarrow & & \downarrow & & \downarrow \sim \\ \{0\} & \xrightarrow{i} & X & \xleftarrow{j} & U \end{array} \qquad (p\colon \tilde{X} \dashrightarrow \tilde{X}^{\mathbb{G}_m})$$

We easily see that $j^*j_*\underline{\mathbb{C}}_U = \underline{\mathbb{C}}_U$ and hence it suffices to compute $i^*j_*\underline{\mathbb{C}}_U$. By tracing through the above diagram and using proper base change for the left-hand square, one easily sees that

$$i^*j_*\underline{\mathbb{C}}_U = R\Gamma \circ \tilde{i}^* \circ \tilde{j}_*\underline{\mathbb{C}}_U = R\Gamma \circ p_* \circ \tilde{j}_*\underline{\mathbb{C}}_U$$

where the second equality follows from our result on $\mathbb{G}_m$-localizations.
Now the fiber of $p \circ \tilde{j}$ over a point $[\ell] \in \mathbb{P}^1 \simeq \tilde{X}^{\mathbb{G}_m}$ is the set of all non-zero matrices whose kernel contain the line ℓ.
One easily sees that this set is homeomorphic to $\mathbb{C}^2 \setminus \{0\} \simeq S^3$.
Hence $p \circ \tilde{j}$ defines a S^3 bundle over $\mathbb{P}^1 \simeq S^2$, the Serre spectral sequence associated to this bundle has E_2-page

$\mathbb{C}$	0	$\mathbb{C}$
0	0	0
0	0	0
$\mathbb{C}$	0	$\mathbb{C}$

For degree reasons, this sequence collapses and we obtain the following table

$j_*\underline{\mathbb{C}}_U$	U	$\{0\}$
5	0	$\mathbb{C}$
4	0	0
3	0	$\mathbb{C}$
2	0	$\mathbb{C}$
1	0	0
0	$\mathbb{C}$	$\mathbb{C}$

Cohomology of GL_2 method

The natural $\mathbb{G}_m$-action on X given by scaling has fixed points 0, let $p\colon X \to X^{\mathbb{G}_m} = \{0\}$ be the attractor map. We see that

$$i^* j_* \underline{\mathbb{C}}_U = (p \circ j)_* \underline{\mathbb{C}}_U = R\Gamma(U, \underline{\mathbb{C}}_U)$$

Hence, the stalk at 0 is given by the singular cohomology of U, notice that there is a decomposition $U \coprod GL_2 = \mathbb{C}^4 \setminus 0$ yields a triangle

$$i_{U*} i_U^! \underline{\mathbb{C}}_{\mathbb{C}^4 \setminus 0} \to \underline{\mathbb{C}}_{\mathbb{C}^4 \setminus 0} \to j_{GL_2 *} j_{GL_2}^* \underline{\mathbb{C}}_{\mathbb{C}^4 \setminus 0} \to \cdot$$

where i_U and j_{GL_2} are the obvious closed resp. open immersions.
Since $U \hookrightarrow \mathbb{C}^4 \setminus 0$ is a closed immersion and both varieties are smooth, the shriek pullback of the dualizing complex is the dualizing complex ie. $i_U^! \underline{\mathbb{C}}_{\mathbb{C}^4 \setminus 0} = i_u^* \underline{\mathbb{C}}_U[-2]$ and thus, taking global sections one obtains the triangle

$$R\Gamma(U, \underline{\mathbb{C}}_U)[-2] \to R\Gamma(\mathbb{C}^4 \setminus 0, \underline{\mathbb{C}}_{\mathbb{C}^4 \setminus 0}) \to R\Gamma(GL_2, \underline{\mathbb{C}}_{GL_2} \to \cdot$$

Using that $GL_2 \simeq S^1 \times S^3$, one can compute $R^i\Gamma(U, \underline{\mathbb{C}}_U) = H^i_{sing}(U, \mathbb{C})$ and hence the stalk at 0 which yields the same decomposition.

$j_*\underline{\mathbb{C}}_U$	U	$\{0\}$
5	0	$\mathbb{C}$
4	0	0
3	0	$\mathbb{C}$
2	0	$\mathbb{C}$
1	0	0
0	$\mathbb{C}$	$\mathbb{C}$

$\omega_X = \mathbb{D}(\underline{\mathbb{C}}_X)$

First, we note that

$$j^*\mathbb{D}(\underline{\mathbb{C}}_X) = \mathbb{D}(j^!\underline{\mathbb{C}}_X) = \mathbb{D}(\underline{\mathbb{C}}_U) = \underline{\mathbb{C}}_U[6]$$

To compute the stalk at 0*, we will use that there is a triangle*

$$\tau^{\leq 0} j_*\underline{\mathbb{C}}_U \to j_*\underline{\mathbb{C}}_U \to \tau^{\geq 1} j_*\underline{\mathbb{C}}_U$$

and that $\tau^{\leq 0} j_*\underline{\mathbb{C}}_U = \underline{\mathbb{C}}_X$ *by the above computation.*
Applying Verdier duality we get a triangle

$$\mathbb{D}(\tau^{\geq 1} j_*\underline{\mathbb{C}}_U) \to \mathbb{D}(j_*\underline{\mathbb{C}}_U) \to \omega_X \to \cdot$$

Pulling back along i *this yields*

$$\mathbb{D}(i^!\tau^{\geq 1} j_*\underline{\mathbb{C}}_U) \to \mathbb{D}(\underbrace{i^! j_*\underline{\mathbb{C}}_U}_{=0}) \to i^*\omega_X \to \cdot$$

where we used [Ach21, theorem 1.3.10].
Thus, it suffices to compute $i^!\tau^{\geq 1} j_*\underline{\mathbb{C}}_U$*, notice that we can write*

$$\tau^{\geq 1} j_*\underline{\mathbb{C}}_U = i_*(\underbrace{i^*\tau^{\geq 1} j_*\underline{\mathbb{C}}_U}_{:=\mathcal{G}}) = i_*\mathcal{G} = i_!\mathcal{G}.$$

$$i^!\tau^{\geq 1} j_*\underline{\mathbb{C}}_U = i^! i_!\mathcal{G} = p_! i_!\mathcal{G} = \mathcal{G}$$

where we again used $\mathbb{G}_m$*-localization. Hence, we see that* $i^*\omega_X = \mathbb{D}(\mathcal{G})$ *and we obtain the table*

ω_X	*U*	{0}
-3	0	$\mathbb{C}$
-4	0	$\mathbb{C}$
-5	0	0
-6	$\mathbb{C}$	$\mathbb{C}$

$IC(X)$

To compute this, notice that the map $\tilde{X} \to X$ *is semismall, hence by the above theorem we see that*

$$j_{!*}\underline{\mathbb{C}}_U = j_{!*} q_*\underline{\mathbb{C}}_{q^{-1}(U)} = q_*\underline{\mathbb{C}}_{\tilde{X}}.$$

We know how to compute the expression on the right, see the first part of this example, and this yields

-	U	$\{0\}$
2	0	$\mathbb{C}$
1	0	0
0	$\mathbb{C}$	$\mathbb{C}$

5 Non-split Filtrations on Perverse Sheaves

Recall that last talk we proved that every perverse sheaves admits a filtration by IC sheaves and that these are precisely the simple objects of the category of perverse sheaves.
In this last part we give an example of two such filtrations on a perverse sheaf that are not just direct sums (ie. we show that in general the category of perverse sheaves is not semisimple).
We fix the usual stratification $\{0\} \coprod \mathbb{A}^1 \setminus 0 = \mathbb{A}^1$, let i be the inclusion of the closed stratum and j the inclusion of the complementary open stratum.
We will describe a decomposition into two IC sheaves of the perverse sheaves $j_*\underline{\mathbb{C}}[1]$ and $j_!\underline{\mathbb{C}}[1]$.

We start by computing the filtration on $j_!\mathbb{C}[1]$.
Notice that there is an obvious epimorphism $\alpha\colon j_!\underline{\mathbb{C}}[1] \to j_{!*}\underline{\mathbb{C}}[1]$ and we claim that the kernel is $IC(\{0\} \subset X)$.
Denote the kernel K, it suffices to show that it satisfies the characterization of [Ach21, lemma 3.3.4].
Clearly, away from 0, α is an isomorphism and hence $K|_U = 0$. It follows that the support of K is $\{0\}$ or empty. Now, notice that the kernel of α is the same as the kernel of the map $j_!\underline{\mathbb{C}}[1] \to j_*\underline{\mathbb{C}}[1]$.
Using that j_* is exact for the usual t-structure and that $i^*j_! = 0$ (see [Ach21, theorem 1.3.10 a]), we find that $\mathbb{C}_{\{0\}} = i^*j_*\underline{\mathbb{C}} \simeq i^*K$. As $\{0\}$ is a closed subset the other conditions of the lemma are obsolete and there is a short exact sequence

$$0 \to IC(\{0\} \subset \mathbb{A}^1) \to j_!\underline{\mathbb{C}}[1] \to IC(U \subset \mathbb{A}^1) \to 0.$$

Applying Verdier duality to this sequence one obtains a similar filtration on $j_*\underline{\mathbb{C}}[1]$ which is

$$0 \to IC(U \subset \mathbb{A}^1) \to j_*\underline{\mathbb{C}}[1] \to IC(\{0\} \subset \mathbb{A}^1) \to 0.$$

By comparing these filtrations with the examples obtained in section 1, we see that the filtrations on $j_!\underline{\mathbb{C}}[1]$ and $j_*\underline{\mathbb{C}}[1]$ could not have been split.

Another way of seeing this is that there are no non-zero maps $\mathrm{IC}(\{0\}) = i_* \underline{\mathbb{C}}_{\{0\}} \to j_* \underline{\mathbb{C}}[1]$ and hence the filtrations could not have been split.

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