Intersection Cohomology Complexes

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Contents

0 Addendum to last talk

We state some exact properties of special functors with respect to the perverse t-structure. The proofs are left as an exercise or can be found in [1].

Proposition 0.1. *[1][3.1.11]*
The Verdier Duality functor $\mathbb{D}\colon D^b_c(X)^{op} \to D^b_c(X)$ *is* t*-exact for the perverse* t*-structure.*

Proposition 0.2. *([1][3.1.12]*
Let $f\colon X \to Y$ *be a finite morphism. The functor* $f_*\colon D^b_c(X) \to D^b_c(Y)$ *is* t*-exact for the perverse* t*-structure.*

Proposition 0.3. *([1][3.2.2])*
Let X *be a (complex) variety and* $\mathcal{L}$ *a local system of finite type on* X*. The functors*

$$(-) \otimes \mathcal{L} \text{ and. } R\mathcal{H}om(\mathcal{L}, -)\colon D^b_c(X) \to D^b_c(X)$$

are t*-exact for the perverse* t*-structure.*
Note that in the general relative setting (non-field coefficients) one only has left resp. right t*-exactness.*

Similarly we have

Proposition 0.4. *([1][3.2.5])*
The exterior tensor product $\boxtimes$ *is* t*-exact for the perverse* t*-structure. In the general relative setting (non field coefficients) it only preserves the* ≤ 0 *part of the perverse* t*-structure.*

1 Intersection Cohomology Complexes and properties

1.1 Definitions

In the last talks we saw that the heart of the pervese t-structure on $D_c^b(X)$, called *the category of Perverse sheaves* $\mathrm{Perv}(X)$ is an abelian category. It thus makes sense to ask about subobjects of an object $\mathcal{F} \in \mathrm{Perv}(X)$ and to talk about its simple objects (i.e. objects containing no proper subobjects). We will be able to give a fairly concrete and geometric description of these objects following [1][Ch.3.3], which will later be called *Intersection Cohomology Complexes* or *Intersection Cohomology Sheaves*.
The main tool in this talk is the functor constructed as follow:
Let $h\colon Y \hookrightarrow X$ be a locally closed embedding of varieties. We have studied the functors $h_!$ as well as h_* and saw that there is a natural transformation $h_! \to h_*$. This yields the following definition:

Definition 1.1. *Let $h\colon Y \hookrightarrow X$ be a locally closed embedding of complex varieties. The* Intermediate-extension functor *is the functor*

$$h_{!*}\colon\ Perv(Y, \mathbb{C}) \to Perv(X, \mathbb{C}),$$

given by

$$h_{!*}(\mathcal{F}) := im({}^pH^0(h_!\mathcal{F}) \to {}^pH^0(h_*\mathcal{F})).$$

Remark 1.2. *Note that if h is proper (e.g. $Y \subset X$ is a closed subvariety), then $h_! \simeq h_*$ and thus $h_{!*}(\mathcal{F}) = {}^pH^0(h_*\mathcal{F})$. Thus on the level of $Perv(X)$, $h_{!*} \simeq h_*$.*

We are now already ready to define intersection cohomology complexes. Note that in this talk we will always be in the situation of X being a complex variety.

Definition 1.3. *Let X be a complex variety. Let $h\colon Y \hookrightarrow X$ be a smooth, connected, locally closed subvariety (think $Y = X_{sm}$ the smooth locus) and let $\mathcal{L} \in Loc^{ft}(Y)$. The* intersection cohomology complex *associated to $(Y, \mathcal{L})$ is the perverse sheaf*

$$IC(Y, \mathcal{L}) := h_{!*}(\mathcal{L}[\dim Y]).$$

In the case of $Y = X_{sm}$ and $\mathcal{L} = \underline{\mathbb{C}}_Y$, one writes $IC(X; \mathbb{C})$ instead of $IC(Y, \mathcal{L})$ and calls it the intersection cohomology complex *of X.*
The hypercohomology of $IC(X; \mathbb{C})[-\dim X]$ is denoted by

$$\boldsymbol{IH}^k(X; \mathbb{C}) := \mathbb{H}^{k-\dim X}(X, IC(X; \mathbb{C})),$$

and we call it the intersection cohomology *of X.*

Remark 1.4. *Recall that a perverse sheaf has no hypercohomology below $-\dim X$, so in particular $IC(X; \mathbb{C}) \in D_c^b(X)^{\geq -\dim X}$ and thus the intersection cohomology $\boldsymbol{IH}^\bullet$ lives in nonnegative degrees.*

1.2 Properties

We now come to study properties of $h_{!*}$ that will enable us to classify intersection cohoomology complexes as the simple objects of $\mathrm{Perv}(X)$. For this we first come to the following observation (Exercise 3.1.6 in Achar) whose prove we will only sketch

Proposition 1.5. *Let $h\colon Y \hookrightarrow X$ be a locally closed embedding. Let $\mathcal{F}$ be a perverse sheaf on Y and let $\mathcal{G}$ be a perverse sheaf on X that is supported on $\overline{Y} \setminus Y$. Then*

1. *$Hom({}^pH^0(h_!\mathcal{F}), \mathcal{G}) = 0$*
2. *$Hom(\mathcal{G}, {}^pH^0(h_*\mathcal{F})) = 0$.*

Proof. (Sketch)
We choose a constructible stratification for $\tilde{\mathcal{F}} := {}^pH^0(h_!\mathcal{F})$ and $\mathcal{G}$ and induct on the size of the stratification. Denoting $i\colon X_t \hookrightarrow X \hookleftarrow X \setminus X_t\colon j$ the complementary embeddings corresponding to some stratum X_t, we use the LES

$$\mathrm{Hom}(i^*\tilde{\mathcal{F}}, i^!\mathcal{G}) \to \mathrm{Hom}(\tilde{\mathcal{F}}, \mathcal{G}) \to \mathrm{Hom}(j^*\tilde{\mathcal{F}}, j^*\mathcal{G}) \to \dots.$$

Now the restriction to the open $X \setminus X_t$ falls under the assumption of the induction so the third term in the sequence is 0 ($X \setminus X_t$ is stratified by a smaller Stratification). For the first term in the sequence we

use the assumption on the support of $\mathcal{G}$ and the fact $\tilde{\mathcal{F}} \in D^b_c(X)^{\leq 0}$ (implied by perversity) to show that $i^*\tilde{\mathcal{F}}$ and $i^!\mathcal{G}$ live in different parts of the perverse t-structure and thus have no morphisms.
The second statement follows analogously.
For more details, see [2][Lemma 2.11] □

Lemma 1.6. *Let $h\colon Y \hookrightarrow X$ be a locally closed embedding.*

1. *For $\mathcal{F} \in Perv(Y)$, there is a natural isomorphism $h^*h_{!*}\mathcal{F} \simeq \mathcal{F}$.*
2. *For $\mathcal{F} \in Perv(Y)$, the object $h_{!*}\mathcal{F}$ has no nonzero subobjects or quotients supported on $\overline{Y} \setminus Y$.*

Proof. Obviously $h_!\mathcal{F}$ and $h_*\mathcal{F}$ are both supported on $\overline{Y}$, thus we can factor h through an open embedding $Y \hookrightarrow \overline{Y} \hookrightarrow X$ and assume wlog $X = \overline{Y}$ and h open. For open embeddings, we know that h^* is t-exact for the perverse t-structure, thus we have

$$h^*h_{!*}\mathcal{F} = h^*\mathrm{im}({}^pH^0(h_!\mathcal{F}) \to {}^pH^0(h_*\mathcal{F})) \simeq \mathrm{im}(h^{*p}H^0(h_!\mathcal{F}) \to h^{*p}H^0(h_*\mathcal{F})).$$

Now again using t-exactness to commute h^* and ${}^pH^0$ we get

$$h^*h_{!*}\mathcal{F} \simeq \mathrm{im}({}^pH^0(h^*h_!\mathcal{F}) \to {}^pH^0(h^*h_*\mathcal{F})) \simeq \mathrm{im}(\mathcal{F} \to \mathcal{F}) = \mathcal{F}.$$

In the second to last equality we use $h^*h_! \simeq h^*h_* \simeq \mathrm{id}$ and the fact that $\mathcal{F}$ is perverse.
Now let $Z \coloneqq X \setminus Y$. If $\mathcal{G} \subset h_{!*}\mathcal{F}$ with $\mathrm{supp}(\mathcal{G}) \subset Z$, then via $h_{!*}\mathcal{F} \hookrightarrow {}^pH^0(h_*\mathcal{F})$, we would have $\mathcal{G} \subset {}^pH^0(h_*\mathcal{F})$. But $\mathrm{Hom}(\mathcal{G}, {}^pH^0(h_*\mathcal{F})) = 0$ for perverse $\mathcal{G}$ supported on Z by 1.5, so $\mathcal{G} = 0$. Now let $\mathcal{G}$ be a quotient of $h_{!*}\mathcal{F}$, then via ${}^pH^0(h_!\mathcal{F}) \twoheadrightarrow h_{!*}\mathcal{F}$ we regard $\mathcal{G}$ as a quotient of ${}^pH^0(h_!\mathcal{F})$. Now similarly $\mathrm{Hom}({}^pH^0(h_!\mathcal{F}), \mathcal{G}) = 0$ and thus $\mathcal{G} = 0$ and we are done. □

We see that the intermediate extension of a perverse sheaf $\mathcal{F}$ is supported on $\overline{Y}$, restricts back to $\mathcal{F}$ on Y and has no subobjects supported on the boundary. In fact, $h_{!*}\mathcal{F}$ turns out to be unique with respect to this property. Indeed we have

Lemma 1.7. *Let $h\colon Y \hookrightarrow X$ be a locally closed embedding. The intermediate extension functor $h_{!*}\colon Perv(Y) \to Perv(X)$ is fully faithful. For $\mathcal{F} \in Perv(Y)$, the object $h_{!*}\mathcal{F}$ is the unique perverse sheaf on X with the following properties:*

1. *It is supported on $\overline{Y}$.*
2. *Its restriction to Y is isomorphic to $\mathcal{F}$ ($h^*h_{!*}\mathcal{F} \simeq \mathcal{F}$).*
3. *It has no nonzero subobjects or quotients supported on the boundary $\overline{Y} \setminus Y$.*

Proof. 1. (*Fully faithfulness*):
Again we may assume $X = \overline{Y}$ and h open embedding. Denote by $i\colon Z \hookrightarrow X$ the complementary closed immersion.
To show that $h_{!*}\colon \mathrm{Perv}(Y) \to \mathrm{Perv}(X)$ is fully faithful, we show that it is part of an equivalence.
By 1.6 above, a candidate for its inverse will be h^*.
In this spirit, consider the full subcategory $\mathrm{Perv}^0(X) \subset \mathrm{Perv}(X)$ conisting of perverse sheaves with no nonzero subobjects or quotients on Z. Note that we clearly have

$$h^*\colon \mathrm{Perv}^0(\overline{Y}) \rightleftarrows \mathrm{Perv}(Y)\colon h_{!*}.$$

Now if we show that this is an equivalence of categories, we are done. By 1.6 we already know, that if h^* is part of an equivalence, $h_{!*}$ is its inverse.
Let $\mathcal{F} \in \mathrm{Perv}^0(X)$, consider the standard exact triangle

$$h_!h^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F}.$$

Now as $\mathcal{F}$ is perverse, we have by the t-exactness properties of our functors, that all three terms in this triangle are in ${}^pD^b_c(X)^{\leq 0}$ so we get a long exact perverse cohomology sequence

$$\cdots \to {}^pH^0(h_!h^*\mathcal{F}) \to {}^pH^0(\mathcal{F}) \to {}^pH^0(i_*i^*\mathcal{F}) \to 0.$$

In particular ${}^pH^0(\mathcal{F}) \to {}^pH^0(i_*i^*\mathcal{F})$ is surjective. Now as $i_*i^*\mathcal{F}$ and thus ${}^pH^0(i_*i^*\mathcal{F})$ is supported on Z and its a quotient of $\mathcal{F} = {}^pH^0(\mathcal{F}) \in \mathrm{Perv}^0(X)$, we have ${}^pH^0(i_*i^*\mathcal{F}) = 0$. Thus, equivalently, $i_*i^*\mathcal{F} \in {}^pD^b_c(X)^{\leq -1}$. Now, as i_* is t-exact and fully faithful, we have

$$i^*\mathcal{F} \in {}^pD^b_c(Z)^{\leq -1},$$

and similarly

$$i^!\mathcal{F} \in {}^pD^b_c(Z)^{\geq 1}.$$

Now let $\mathcal{G} \in \mathrm{Perv}^0(X)$, then, as $i^*\mathcal{F}$ and $i^!\mathcal{G}$ live in disjoint cohomological degree, we obtain a long exact sequence (cf. Julius' talk):

$$\ldots \longrightarrow \mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}[-1]) \longrightarrow \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G})$$

$$\mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}) \longrightarrow \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}[1]) \longrightarrow \ldots$$

Now in $\mathrm{Perv}(X)$ by the definition of the perverse t-structure and the fact that i^* and $i^!$ land ind different cohomological degrees, this degenerates to

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}).$$

I.e. $h^*\colon \mathrm{Perv}^0(\overline{Y}) \to \mathrm{Perv}(Y)$ is fully faithful. Again, 1.6 shows that it is essentially surjective with inverse $h_{!*}$ and thus also $h_{!*}$ is fully faithful.

2. and 3.
Both the properties are clearly satisfied by $h_{!*}$ and uniqueness follows from exhibiting $h_{!*}$ as part of an equivalence.

□

Remark 1.8. *The above proof shows yet another, more cohomological, description of the intermediate extension of a perverse sheaf. In the above notation one can see that $\mathcal{F} \simeq h_{!*}\mathcal{F}' \Leftrightarrow \mathcal{F}$ is supported on $\overline{Y}$, restricts to $\mathcal{F}'$ on Y and satisfies $i^!\mathcal{F} \in {}^pD^b_c(Z)^{\geq 1}$ and $i^* \in {}^pD^b_c(Z)^{\leq -1}$.*

We now have a way to associated a unique extension of $\mathcal{F} \in \mathrm{Perv}(Y)$ to its closure, which is simple on the boundary. It is thus natural to ask for a decomposition of $\mathcal{F}$ into complexes supported on a closed subsets, which is maximal in some sense. The following result should not be surprising.

Lemma 1.9. *Let $\mathcal{F} \in Perv(X)$, let $i\colon Z \hookrightarrow X$ be a closed subvariety.*

1. *The natural map ${}^pH^0(i_*i^!\mathcal{F}) \to \mathcal{F}$ is injective and universal among subobjects of $\mathcal{F}$ supported on Z. In particular, for all $\phi\colon \mathcal{G} \hookrightarrow \mathcal{F}$, with $\mathcal{G}$ perverse and supported on Z, there exists a unique ϕ' such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{G} \arrow[dr, hook, dashed, "\exists!\phi'"'] \arrow[drr, hook, "\phi"] & & \\ & {}^pH^0(i_*i^!\mathcal{F}) \arrow[r, hook] & \mathcal{F} \end{tikzcd}$$

.

2. *The natural map $\mathcal{F} \to {}^pH^0(i_*i^*\mathcal{F})$ is surjective and universal among quotients of $\mathcal{F}$ supported on Z. In particular, for all $\phi\colon \mathcal{F} \twoheadrightarrow \mathcal{G}$ with $\mathcal{G}$ perverse and supported on Z, there exists a unique ϕ' such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{F} \arrow[r, two heads] \arrow[drr, two heads, "\phi"'] & {}^pH^0(i_*i^*\mathcal{F}) \arrow[dr, two heads, dashed, "\exists!\phi'"] & \\ & & \mathcal{G} \end{tikzcd}$$

.

Proof. 1. Consider again the standard triangle associated to $i\colon Z \hookrightarrow X \hookleftarrow U\colon j$:

$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to$$

now, we know how the functors $i_*, i^!, j_*, j^*$ behave with respect to the perverse t-structure, thus all three terms lie in ${}^pD^b_c(X)^{\geq 0}$. We obtain the long exact sequence associated to perverse cohomology

$$0 \to {}^pH^0(i_*i^!\mathcal{F}) \to {}^pH^0(\mathcal{F}) \simeq \mathcal{F} \to {}^pH^0(j_*j^*\mathcal{F}) \to {}^pH^1(i_*i^!\mathcal{F}) \to \ldots,$$

thus the desired map ${}^pH^0(i_*i^!\mathcal{F}) \to \mathcal{F}$ is injective.
Now let $\mathcal{G}$ be a peverse sheaf supported on Z. Applying $\mathrm{Hom}(\mathcal{G}, -)$ to the above sequence, yields

$$\ldots \longrightarrow \mathrm{Hom}(\mathcal{G}, j_*j^*\mathcal{F}[-1]) \longrightarrow \mathrm{Hom}(\mathcal{G}, i_*i^!\mathcal{F}) \longrightarrow \mathrm{Hom}(\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}(\mathcal{G}, j_*j^*\mathcal{F}) \longrightarrow \ldots$$

Now as $\mathcal{G}$ is supported on Z, we have $j^*\mathcal{G} = 0$ and thus by adjunction the outer terms of the above sequence vanish, yielding $\mathrm{Hom}(\mathcal{G}, i_*i^!\mathcal{F}) \simeq \mathrm{Hom}(\mathcal{G}, \mathcal{F})$. Thus $\phi \in \mathrm{Hom}(\mathcal{G}, \mathcal{F})$ factors uniquely as $\mathcal{G} \to i_*i^!\mathcal{F} \to \mathcal{F}$ and as $\mathcal{G}$ and $\mathcal{F}$ are perverse, ${}^pH^0(-)$ of this factorization yields exactly the statement.

2. For the second statement, replace ${}^pD^b_c(X)^{\geq 0}$ by ${}^pD^b_c(X)^{\leq 0}$ and $\mathrm{Hom}(\mathcal{G}, -)$ by $\mathrm{Hom}(-, \mathcal{G})$ and the proof is almost the same.

□

Corollary 1.10. *The cokernel $p\colon \mathcal{F} \twoheadrightarrow \mathcal{Q}$ of the natural map ${}^pH^0(i_*i^!\mathcal{F}) \hookrightarrow \mathcal{F}$ has no subobjects supported on Z.*
The dual statement holds aswell.

Proof. To shorten the notation, we write $\mathcal{H} := {}^pH^0(i_*i^!\mathcal{F})$.
Let $\mathcal{Q}' \subset \mathcal{Q}$ be a subobject of $\mathcal{Q}$ supported on Z. Consider the diagram

$$\begin{tikzcd}
\mathcal{H} \cap p^{-1}\mathcal{Q}' \arrow[r, hook] \arrow[d, hook] & p^{-1}\mathcal{Q}' \arrow[r, two heads] \arrow[d, hook] & \mathcal{Q}' \arrow[d, hook] \\
\mathcal{H} \arrow[r, hook] & \mathcal{F} \arrow[r, two heads, "p"] & \mathcal{Q}.
\end{tikzcd}$$

Now the above row yields $\mathrm{supp}(p^{-1}\mathcal{Q}') = \mathrm{supp}\left(\mathcal{H} \cap p^{-1}\mathcal{Q}'\right) \cup \mathrm{supp}(\mathcal{Q}') \subset Z$ by assumption and thus by the universal property $p^{-1}\mathcal{Q}' \subset \mathcal{H}$. By commutativity of the diagram, we thus have $p^{-1}\mathcal{Q}' \subset \ker(p)$ and thus $\mathcal{Q}' = 0$.
Similarly for the dual statement

□

This immediately yields a decomposition result for perverse sheaves along a closed subvariety. We have now classified the maximal subobject (resp. quotient) supported on a closed subvariety, of a perverse sheaf $\mathcal{F}$. Thus if $\mathcal{F}$ itself has no subobject (resp. quotient) on Z, by 1.10, the kernel (resp. cokernel) of the natural map $\mathcal{F} \twoheadrightarrow {}^pH^0(i_*i^*\mathcal{F})$ (resp. ${}^pH^0(i_*i^!\mathcal{F}) \hookrightarrow \mathcal{F}$) has no quotient (resp. subobject) supported on Z. Now as this object is itself a subobject (resp. quotient) of $\mathcal{F}$ it has no subobject (resp. quotient) supported on Z either. Thus by 1.7 the kernel (resp. cokernel) is just given by $j_{!*}(\mathcal{F}|_U)$.
Thus we have

Lemma 1.11. *Let X be an irreducible variety. Let $j\colon U \hookrightarrow X \hookleftarrow Z\colon i$ be complementary embeddings with U open and Z closed. Let $\mathcal{F} \in Perv(X)$.*

1. *If $\mathcal{F}$ has no quotient supported on Z, then there is a natural SES:*

$$0 \to {}^pH^0(i_*i^!\mathcal{F}) \to \mathcal{F} \to j_{!*}(\mathcal{F}|_U) \to 0$$

2. *If $\mathcal{F}$ has no subobject supported on Z, then there is a natural SES:*

$$0 \to j_{!*}(\mathcal{F}|_U) \to \mathcal{F} \to {}^pH^0(i_*i^*\mathcal{F}) \to 0$$

Proof. Above discussion and 1.7. □

1.8 together with a result from last talk, lets us give a geometric description of intersection cohomology sheaves.

Lemma 1.12. *Let $\mathcal{F} \in Perv(X)$. Let $(X_s)_{s\in\mathcal{S}}$ be a stratification with respect to which both $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible. Let $u \in \mathcal{S}$ and let $\mathcal{L} \in Loc^{ft}(X_u)$. TFAE:*

1. *$\mathcal{F} \simeq IC(X_u, \mathcal{L})$.*
2. *$supp(\mathcal{F}) = \overline{X_u}$. Moreover, $\mathcal{F}|_{X_u} \simeq \mathcal{L}[dim\ X_u]$ and for each stratum $X_t \subset \overline{X_u} \setminus X_u$ we have*

$$j_t^*\mathcal{F} \in D^b_{locft}(X_t)^{\leq -dim\ X_t - 1} \text{ and } j_t^!\mathcal{F} \in D^b_{locft}(X_t)^{\geq -dim\ X_t + 1}.$$

This yields a description of IC-sheaves, as perverse sheaves satisfying a strict lower triangularity condition in cohomology. This can be used to provide easy counter examples to perverse sheaves that arent IC-sheaves (i.e. if the cohomology is actually supported on the diagonal). For an example, take $j\colon \mathbb{C}^\times \hookrightarrow \mathbb{C}$ and consider $j_*\underline{\mathbb{C}}_{\mathbb{C}^\times}$ under the stratification $\mathbb{C} = \mathbb{C}^\times \amalg \{0\}$ as in Example 1.15 of last talk.

Example 1.13. *Let X be smooth and $\mathcal{L}[dim\ X] \in Loc^{ft}(X)$, in particular $\mathcal{L} \in Perv(X)$. Then*

$$IC(X, \mathcal{L}) = id_{!*}(\mathcal{L}[dim\ X]) = \mathcal{L}[dim\ X],$$

as id is proper and thus $id_{!*} = id_* = id$ and as $\mathcal{L}[dim\ X]$ is perverse, ${}^pH^0$ also restricts to the identity on $\mathcal{L}[dim\ X]$.
Moreover, in view of our next result, one may compute the Verdier dual of this intersection cohomology complex and observe

$$\mathbb{D}\left(IC(X, \mathcal{L})\right) = \mathbb{D}(\mathcal{L}[n]) = \left(\mathcal{L}[n]\right)^\vee [2n] = \mathcal{L}^\vee[n] \simeq IC(X, \mathcal{L}^\vee).$$

Now to see how intersection cohomology can bee seen as a tool to repair Poincaré duality, we have the following

Lemma 1.14. *Let $h\colon Y \hookrightarrow X$ be a smooth, connected, locally closed subvariety (note that importantly, X is not assumed to be smooth). Let $\mathcal{L} \in Loc^{ft}(Y)$. Then there exists a natural isomorphism*

$$\mathbb{D}(IC(Y, \mathcal{L})) \simeq IC(Y, \mathcal{L}^\vee).$$

Remark 1.15. *We can interpret this result as a sort of Poincaré duality for singular varieties ($X_{sm} \hookrightarrow X$). Note that for $\mathcal{L} = \underline{\mathbb{C}}[-n]$ we obtain $IC(Y, \mathcal{L}) = \underline{\mathbb{C}}$ in the smooth case. Then the statement of the Lemma is exactly $\omega_X = \mathbb{D}\underline{\mathbb{C}} = \underline{\mathbb{C}}[2n]$, which yields classical Poincaré duality (see Tim's talk).*
In general one can replace $\underline{\mathbb{C}}$ by any Intersection cohomology sheaf and obtain

$$\mathbb{D}R\Gamma\left(IC(Y, \mathcal{L})\right) = R\Gamma_c\left(\mathbb{D}(IC(Y, \mathcal{L}))\right) = R\Gamma(IC(X, \mathcal{L}^\vee)),$$

which yields Poincaré duality for intersection cohomology, after passing to H^k.
For smooth varieties, intersection cohomology sheaves are simply shifted local systems. Thus the extent to which $IC(X, \mathcal{L})$ fails to be a local system can be understood to measure the singularity of X in some sense and 1.14 can be thought of Poincaré duality for singular varieties.

Proof. Clearly $\mathbb{D}(IC(Y, \mathcal{L}))$ is supported on $\overline{Y}$ and Verdier Duality commutes with restriction, so

$$\mathbb{D}(IC(Y, \mathcal{L}))|_Y \simeq \mathbb{D}(IC(Y, \mathcal{L})|_Y) \simeq \mathbb{D}(\mathcal{L}[n]) \simeq \mathcal{L}^\vee[n],$$

by 1.13. We now show, that $\mathbb{D}(IC(Y, \mathcal{L}))$ satisfies the universal property of $IC(Y, \mathcal{L}^\vee)$. We already know that is is supported on $\overline{Y}$ and restricts to $\mathcal{L}^\vee[n]$ on Y by the equation above, so if we show that $\mathcal{G} := \mathbb{D}(IC(Y, \mathcal{L}))$ has no nonzero subobjects (resp. quotients) on the boundary, we have $\mathcal{G} \simeq h_{!*}\mathcal{L}^\vee[n] = IC(Y, \mathcal{L}^\vee)$ by 1.7.
Assume $\mathcal{F} \subset \mathcal{G}$ is a subobject supported on $\overline{Y} \setminus Y$. Now we know that $\mathbb{D}$ is contravariant exact, so we obtain a quotient $\mathbb{D}\mathcal{G} = IC(Y, \mathcal{L}) \twoheadrightarrow \mathbb{D}\mathcal{F}$, but we know that $IC(Y, \mathcal{L})$ has no quotients supported on the boundary, so $\mathbb{D}\mathcal{F} = 0$ and thus $\mathcal{F} = 0$. Similarly one shows that $\mathcal{G}$ has no quotients on $\overline{Y} \setminus Y$ and thus the claim holds. ∎

2 Intersection Cohomology sheaves as simple objects

We wish to establish intersection cohomology sheaves as the simple objects of $\mathrm{Perv}(X)$ to filter any perverse sheaf by such complexes.
The first result is the following

Proposition 2.1. *Let X be a smooth, connected variety, dim $X = n$. $Loc^{ft}(X)[n]$ is a Serre subcategory of Perv(X), i.e. it is closed under extensions, subobjects and quotients.*

Proof. omitted. □

The first step towards characterizing Intersection cohomology sheaves as the simple objects of $\mathrm{Perv}(X)$ is to show that all simple perverse sheaves are of the form $IC(Y, \mathcal{L})$.

Theorem 2.2. *Every perverse sheaf admits a finite filtration such that all subqotients are intersection cohomology complexes.*

Remark 2.3. *Let $\mathcal{F} \in$ Perv(X) be simple. Then the only filtration is the trivial one, with quotient $\mathcal{F}$. Thus by the theorem, $\mathcal{F}$ is an intersection cohomology complex.*

Proof. (of theorem)
We proceed by noetherian induction (induction on the dimension of the support of $\mathcal{F}$).
Let $\mathcal{F} \in \mathrm{Perv}(X)$. Consider $j : U \hookrightarrow X \hookleftarrow Z : i$ be the complementary open and closed embeddings of an admissible stratum U such that $\mathcal{F}|_U = \mathcal{L}[\dim U]$ for some local system $\mathcal{L}$. Consider the short exact sequence

$$0 \to {}^pH^0(i_* i^! \mathcal{F}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

then by 1.10, the quotient $\mathcal{Q}$ has no subobject supported on Z. However the first term in the sequence is supported on Z, thus restriction to U (j^*) yields

$$\mathcal{L}[n] \simeq j^* \mathcal{F} \simeq j^* \mathcal{Q}.$$

Remember the first isomorphism comes from the definition of U as a trivializing stratum.
The natrual SES in 1.11 associated to $\mathcal{Q}$ yields

$$0 \to j_{!*}(j^* \mathcal{Q}) \simeq j_{!*} \mathcal{L}[n] = IC(U, \mathcal{L}) \to \mathcal{Q} \to {}^pH^0(i_* i^* \mathcal{Q}) \to 0.$$

Now by induction hypothesis, ${}^pH^0(i_* i^* \mathcal{Q})$ has a filtration by intersection cohomology complexes (as its supported on Z which has positive codimension) and thus we also get a filtration for $\mathcal{Q}$. Applying this argument to the SES defining $\mathcal{Q}$, we get that $\mathcal{F}$ has such a filtration. □

The highlight is

Theorem 2.4. *Let X be a complex variety*

1. *For a smooth, connected, locally closed subvariety $Y \subset X$ and $\mathcal{L}$ an irreducible local system on Y, the $IC(Y, \mathcal{L})$ is a simple object in Perv(X).*

2. *Every perverse sheaf admits a finite filration such that all the subquotients are simple intersection cohomology complexes.*

3. *The category Perv(X) is artininan and noetherian (i.e. any descending/ascending chain of objects terminates) and every simple object is a simple intersection cohomology complex.*

We can even say more about the structure of these filtrations, i.e. in particular, the intersection cohomology of an extension of local systems is filtered by a 3-step filtration of intersection cohomology complexes.

Lemma 2.5. *Let $h : Y \subset X$ be a smooth, connected, locally closed subvariety. Let $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}'' \to 0$ be an extension of local systems on Y. Then $IC(Y, \mathcal{L})$ admits a 3-step filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 = IC(Y, \mathcal{L}),$$

such that

$$\mathcal{F}_1 \simeq IC(Y, \mathcal{L}') \text{ and } \mathcal{F}_3/\mathcal{F}_2 \simeq IC(Y, \mathcal{L}'')$$

and such that $\mathcal{F}_2/\mathcal{F}_1$ is supported on the boundary $\overline{Y} \setminus Y$.

Proof. We know that intermediate extension takes injective/surjective maps to injective/surjective maps (application of 1.9 to the kernel/cokernel), thus we get natural maps $\phi\colon IC(Y,\mathcal{L}') \hookrightarrow IC(Y,\mathcal{L})$ and $\psi\colon IC(Y,\mathcal{L}) \twoheadrightarrow IC(Y,\mathcal{L}'')$ and hence the complex (!)

$$IC(Y,\mathcal{L}') \xrightarrow{\phi} IC(Y,\mathcal{L}) \xrightarrow{\psi} IC(Y,\mathcal{L}'').$$

Now to obtain the filtration, we set $\mathcal{F}_1 \coloneqq \text{im } \phi$ and $\mathcal{F}_2 \coloneqq \text{ker } \psi$. The above being a complex implies $\mathcal{F}_1 \subset \mathcal{F}_2$. Certainly, now $IC(Y,\mathcal{L})/\mathcal{F}_2 =: \mathcal{F}_3/\mathcal{F}_2 \simeq IC(Y,\mathcal{L}'')$ and $\mathcal{F}_1 \simeq IC(Y,\mathcal{L}')$. Thus it only remains to show that $\text{supp}(\mathcal{F}_2/\mathcal{F}_1) = \overline{Y} \setminus Y$.
Recall that pulling back the above complex to Y, amounts to applying the composition $h^* h_{!*} \circ [n]$ to the extension $\mathcal{L}$, now this is clearly exact, so restriction to Y yields a SES

$$0 \to IC(Y,\mathcal{L}')|_Y \to IC(Y,\mathcal{L})|_Y \to IC(Y,\mathcal{L}'')|_Y \to 0.$$

Exactness of this sequence, together with our definition of $\mathcal{F}_1$ and $\mathcal{F}_2$ is exactly $(\mathcal{F}_2/\mathcal{F}_1)\,|_Y = 0$. □

We will give a proof of 2.4 without the noetherian property.

Proof. (of 2.4)

1. : Let $\mathcal{F} \subset IC(Y,\mathcal{L})$ be a nonzero subobject of $IC(Y,\mathcal{L})$. By the equivalence of categories for Perverse sheaves on a closed subvariety and Perverse sheaves supported on that subvariety, we can assume $X = \overline{Y}$, such that we have an open immersion $Y \hookrightarrow \overline{Y} = X$. Now $\mathcal{F}$ cannot be supported on the boundary $X \setminus Y$, as $IC(Y,\mathcal{L})$ by 1.7 has no subobjects supported on the boundary. I.e. by assumption $\mathcal{F}|_Y \neq 0$. We have $(\mathcal{F}|_Y)[-\text{dim } Y] \subset \mathcal{L}$ (as $IC(Y,\mathcal{L})$ restricts to $\mathcal{L}[\text{dim } Y]$ on Y) and thus, as $\mathcal{L}$ is irreducible and $\mathcal{F}|_Y$ is nonzero, we have $\mathcal{F}|_Y \simeq \mathcal{L}[\text{dim } Y] \coloneqq \mathcal{L}[n]$. Now as $\mathcal{F}$ has no subobjects supported on $X \setminus Y$ (since $IC(Y,\mathcal{L})$ doesnt), we get by 1.11 an injection $IC(Y,\mathcal{L}) \hookrightarrow \mathcal{F}$. Let
$$\phi\colon IC(Y,\mathcal{L}) \hookrightarrow \mathcal{F} \hookrightarrow IC(Y,\mathcal{L})$$
be the composition, then $\phi|_Y = id|_{\mathcal{L}[n]}$. Now by fully faithfulness of $h_{!*}$ we have $\phi = id|_{IC(Y,\mathcal{L})}$ and thus $\mathcal{F} = IC(Y,\mathcal{L})$.

2. : By 2.2 it is enough to show the statement for Intersection cohomology complexes $IC(Y,\mathcal{L})$. Proceed by noetherian induction.
Choose $y_0 \in Y$, then $\mathcal{L}$ corresponds to a finite dimensional Monodromy, i.e. a finite dimensional representation of $\pi_1(Y, y_0)$. Now this representation admits a filtration into irreducible representations. These irreducible representations in turn yield a filtration of $\mathcal{L}$ with irreducible pieces. Now by 2.5, we obtain a filtration of $IC(Y,\mathcal{L})$ such that all the subquotients are either of the form $IC(Y,\mathcal{L}')$ for some $\mathcal{L}' \subset \mathcal{L}$ or are perverse sheaves supported on the boundary $\overline{Y} \setminus Y$. The induction step thus yields the result.

3. : Follows immediately.

□

We record two notable examples of intersection cohomology sheaves here, however the main interesting examples shall be given in the next talk

Example 2.6. *1. Consider $h\colon 0 \hookrightarrow \mathbb{C}$, then $h_{!*}$ is just taking the skyscraper sheaf at 0, thus*

$$IC(\{0\}, \underline{\mathbb{C}}) \simeq \underline{\mathbb{C}}_{\{0\}}.$$

2. Let X be a smooth variety, let $U \hookrightarrow X$ be a dense open subset and let $\mathcal{L}$ be a local system on X. Then $IC(U, \mathcal{L}|_U) \simeq \mathcal{L}[\dim X]$.
In particular, let $X = \mathbb{P}^1$ and consider the dense open $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, then

$$IC(\mathbb{A}^1, \underline{\mathbb{C}}) \simeq \underline{\mathbb{C}}_{\mathbb{P}^1}[1].$$

References

[1] P. Achar. "Perverse Sheaves and Applications to Representation Theory". In: ().

[2] K. Dykes. "Perverse sheaves learning seminar: Perverse sheaves and intersection homology". In: ().